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Analytic Pro-p Groups

2nd Edition

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Introduction

And the end of all our exploring
Will be to arrive where we started
And know the place for the first time
T.S. Eliot: Little Gidding

The origin of this book was a seminar held at All Souls College, Oxford, in the Spring of 1989. The aim of the seminar was to work through Michel Lazard's paper *Groupes analytiques p-adiques* [L], at least far enough to understand the proof of 'Lubotzky's linearity criterion' (Lubotzky 1988). In fact, Lubotzky's proof combined Lazard's characterisation of p-adic analytic groups with some recent results of Lubotzky and Mann (1987b) on 'powerful' pro-p groups. We found that by reversing the historical order of development, and starting with powerful pro-p groups, we could reconstruct most of the group-theoretic consequences of Lazard's theory without having to introduce any 'analytic' machinery. This was a comforting insight for us (as group theorists), and gave us the confidence to go on and develop what we hope is a fairly straightforward account of the theory of p-adic analytic groups.

The first edition was divided (like Gaul) into three parts. Parts I and II were essentially linear in structure. The point of view in Part I was group-theoretic; in Part II, more machinery was introduced, such as normed algebras and formal power series. Between Parts I and II was an *Interlude* (Chapter 6): this consisted of a series of more or less independent digressions, describing applications of the results to various aspects of group theory.

This second edition is also in three parts. Parts I and II cover the same ground as before, with some additional material; however, the old Chapter 6 has been replaced by several shorter *Interludes*, and a new Part III has appeared. Readers seeking a simple introduction to pro-p

groups and to p-adic analytic groups will still find this in Parts I and II; the four new chapters that constitute Part III deal with a variety of topics that we felt deserved a place, but which are not necessary for an understanding of the basic theory. The book as a whole now gives a fairly comprehensive account of what is known about pro-p groups on the 'smallish' side; 'large' pro-p groups – free products, groups of tree automorphisms and the like – are discussed in the forthcoming book [DSS].

We now outline the contents in more detail. *Part I* is an account of pro-p groups of finite rank. **Chapter 1** is a leisurely introduction to profinite groups and pro-p groups, starting from first principles. **Chapter 2** is about finite p-groups. A finite p-group G is defined to be *powerful* if G/G^p is abelian (if p is odd; the case $p = 2$ is slightly different). The key results established in this chapter are due to Lubotzky and Mann (1987a): (i) *if G is powerful and can be generated by d elements, then every subgroup of G can be generated by d elements;* and (ii) *if G is a p-group and every subgroup of G can be generated by d elements, then G has a powerful normal subgroup of index at most $p^{d(\log(d)+2)}$* (log to the base 2). **Chapter 3** returns to profinite groups. Here the *rank* of a profinite group is defined, in several equivalent ways. Defining a pro-p group G to be powerful if $G/\overline{G^p}$ is abelian (where $\overline{}$ denotes closure, and the proviso regarding $p = 2$ still applies), we deduce from the above results that *a pro-p group has finite rank if and only if it has a powerful finitely generated subgroup of finite index* (Lubotzky and Mann 1987b). This is then used to give several alternative characterisations for pro-p groups of finite rank. **Chapter 4** continues with the deeper investigation of finitely generated powerful pro-p groups. These groups, being 'abelian modulo p', are in many ways rather like abelian groups. In particular, each such group contains a normal subgroup of finite index which is 'uniform'; we shall not define this here, but note that the uniform pro-p groups are among those studied by Lazard under the name 'groupes p-saturables'. Following an exercise in [L], we show that *a uniform pro-p group G has in a natural way the structure of a finitely generated free $\mathbb{Z}_p$-module*, which we denote $(G, +)$ (here $\mathbb{Z}_p$ denotes the ring of p-adic integers). Defining an additional operation 'bracket' on this module, we also indicate how $(G, +)$ can be turned into a Lie algebra L_G over $\mathbb{Z}_p$: this is the first hint of a connection with Lie groups. It follows that the automorphism group of G has a faithful linear representation over $\mathbb{Z}_p$, and hence that G itself is 'linear modulo its centre'. Part I concludes with **Chapter 5**. Here we study the most familiar p-adic analytic group, namely $\mathrm{GL}_d(\mathbb{Z}_p)$, and show quite explicitly that a suitable congruence subgroup is a uniform pro-p group. Together with the results of Chapter 4, this is used to show that *the automorphism group of any pro-p group of finite rank is itself virtually* (i.e. up to finite index) *a pro-p group of finite rank.*

Interlude A is a summary of results, established throughout the book, that characterise the class of pro-p groups of finite rank, and that determine the *dimension* of such a group.

Although *Part II* is headed 'Analytic groups', these do not appear as such until Chapter 8. **Chapter 6** is utilitarian, giving definitions and elementary results about complete normed $\mathbb{Q}_p$-algebras which are needed later; also established here are relevant properties of the *Campbell–Hausdorff formula.* **Chapter 7** forms the backbone of Part II. In it, we show how to define a norm on the group algebra $A = \mathbb{Q}_p[G]$ of a uniform pro-p group G, in a way that respects both the p-adic topology on $\mathbb{Q}_p$ and the pro-p topology on G. (Readers familiar with Chapter 8 of the first edition will note that the construction of the norm has been streamlined, and that the troublesome case $p = 2$ has been tamed by modifying the norm in that case.) The completion $\widehat{A}$ of this algebra with respect to the norm serves two purposes. On the one hand, an argument using the binomial expansion of terms in $\widehat{A}$ is used to show that the group operations in G are given by *analytic functions* with respect to a natural co-ordinate system on G, previously introduced in Chapter 4. On the other hand, $\widehat{A}$ serves as the co-domain for the *logarithm* mapping $\log : G \to \widehat{A}$. We show that the set $\log(G)$ is a $\mathbb{Z}_p$-Lie subalgebra of the commutation Lie algebra on $\widehat{A}$, isomorphic via log to the Lie algebra L_G defined intrinsically in Chapter 4 (in fact, the proof simultaneously establishes that L_G satisfies the Jacobi identity, and that $\log(G)$ is closed with respect to the operation of commutation). An appeal to *Ado's Theorem*, in conjunction with the Campbell–Hausdorff formula, then shows that G has a faithful linear representation over $\mathbb{Z}_p$; it follows that *every pro-p group of finite rank has a faithful linear representation over* $\mathbb{Z}_p$.

The final section of Chapter 7 examines the structure of the completed group algebras $\mathbb{Z}_p[[G]]$ and $\mathbb{F}_p[[G]]$: the associated graded rings are shown to be *polynomial rings over* $\mathbb{F}_p$, which implies that $\mathbb{Z}_p[[G]]$ and $\mathbb{F}_p[[G]]$ are both Noetherian integral domains. These results are not needed for the theory of analytic groups developed in the rest of Part II, but have other important applications (for example in Chapter 12, and in the cohomology theory of p-adic analytic groups, see [DSS]). Under the name of the 'Iwasawa algebra', the completed algebra $\mathbb{Z}_p[[G]]$ plays an important role in the theory of cyclotomic fields (Washington 1982).

The linearity result proved in Chapter 7 is applied in **Interlude B** to establish 'Lubotzky's linearity criterion'. This amounts to the statement that *a finitely generated (abstract) group G has a faithful linear representation over a field of characteristic zero if and only if some pro-p completion of G has finite rank.*

p-adic analytic groups are defined in **Chapter 8**. Although we introduce p-adic manifolds, only a bare minimum of theory is developed (in contrast to Serre (1965), for example, there is no use of differentials). Using the results of Chapters 6 and 7, it is shown that *a pro-p group has a p-adic analytic structure if and only if it has finite rank,* and, more generally, that *every p-adic analytic group has an open subgroup which is a pro-p group of finite rank.* These major results are due to Lazard [L], except that he refers to finitely generated virtually powerful pro-p groups where we have 'pro-p groups of finite rank'.

In **Interlude C** we reprint an announcement by Marcus du Sautoy, where the preceding theory is applied to study the subgroup-growth behaviour of finitely generated groups.

Chapter 9 is concerned with some of the 'global' properties of p-adic analytic groups. The first main result here is that *every continuous homomorphism between p-adic analytic groups is an analytic homomorphism*, from which it follows that *the analytic structure of a p-adic analytic group is determined by its topological group structure.* Next, it is shown that *closed subgroups, quotients and extensions of p-adic analytic groups are again p-adic analytic*; these results now follow quite easily from the corresponding properties of pro-p groups of finite rank. Section 9.4 (which can be read independently of Chapter 8) establishes that *the correspondence $G \leftrightarrow L_G$ is an isomorphism between the category of uniform pro-p groups and the category of 'powerful' Lie algebras over $\mathbb{Z}_p$.* This is used in the final section of the chapter to establish the equivalence of the category of p-adic analytic groups (modulo 'local isomorphism') with the category of finite-dimensional Lie algebras over $\mathbb{Q}_p$.

The first chapter of *Part III*, **Chapter 10**, gives an account of the theory of *pro-p groups of finite coclass* (completing the brief discussion given in §6.4 of the first edition). This beautiful theory is central to the classification of finite p-groups. The chapter can be read directly after Chapter 4, as it depends on the theory of powerful pro-p groups but not on the analytic machinery developed in Part II.

The next two chapters discuss the *dimension subgroup series* in finitely generated pro-p groups. In **Chapter 11** this series is used to derive some more delicate characterisations of pro-p groups of finite rank, originally discovered by Lazard using a different method. **Chapter 12** investigates the *graded restricted Lie algebra* associated with the dimension series; after developing from first principles the necessary theory of restricted Lie algebras, the chapter proves some celebrated theorems of Jennings and of Lazard, about dimension subgroups and about the coefficients in the 'Golod–Shafarevich series' of a finitely generated pro-p group.

The next two *Interludes* give applications of Lazard's theorem on the Golod–Shafarevich series; **Interlude D** is devoted to the *Golod–Shafarevich inequality* in a large class of pro-p groups and abstract groups, while **Interlude E** presents Grigorchuk's theorem about *groups of sub-exponential growth.*

In the Introduction to the first edition, we wrote '... Analytic groups over other fields also deserve consideration; a Lie group in the usual sense (over $\mathbb{R}$ or $\mathbb{C}$) cannot be a pro-p group (except in the trivial, discrete, case), but some extremely interesting pro-p groups arise as analytic groups over fields of characteristic p: for example, suitable congruence subgroups in $\mathrm{SL}_n(\mathbb{F}_p[[t]])$. The theory of such groups poses some exciting challenges: they will have to be faced in a different book.' **Chapter 13** is a first step in facing those challenges. A *pro-p domain* is a commutative, Noetherian complete local integral domain R whose residue field is finite, of characteristic p. We propose a definition for 'groups analytic over R', and generalise one of the main results of Chapter 8 by showing that *every R-analytic group contains an open R-standard subgroup*; here, a group is called *R-standard* if it can be identified with $(\mathfrak{m}^k)^{(n)}$, where $\mathfrak{m}$ is the maximal ideal of R, and the group operation is given by a formal group law with coefficients in R. By their very nature, R-standard groups are pro-p groups. In general they do not have finite rank, but they resemble $\mathbb{Z}_p$-standard groups (which do have finite rank) in interesting ways. These are explored in the later part of the chapter, which establishes some results due to Lubotzky and Shalev (1994).

Appendix A gives a proof of the *Hall–Petrescu* commutator-collection formula, which is used in Chapters 10 and 11. **Appendix B** contains the proofs of some elementary facts about topological groups, used in Chapter 9.

The *Exercises* at the end of each chapter serve two important purposes, beyond the usual one of providing practice in the use of new concepts: some fill gaps in the proofs of the main text, and some lead the reader through the proofs of interesting and/or important results which didn't find a place in the main text, but deserve to be known. They are meant at least to be read, whether or not the reader actually wants to do them!

The brief *Notes* at the end of each chapter represent a very inadequate attempt to give credit where it is due. Because we have done things in our own quixotic way, it has not always been easy to attribute specific results.

To say that the language and much of the theory of profinite groups has passed into 'folklore' does an injustice to the creators; we apologise for this, giving the weak excuse that it's the best we can do. However, it does seem clear that Serre's 'Cohomologie galoisienne' (recently translated as Serre (1997)) was the first appearance in book form of much of the basic theory, particularly as far as pro-p groups are concerned.

The book as a whole should be seen as an exegesis, plus extended commentary, of the first four chapters of Lazard's magisterial work *Groupes analytiques p-adiques* [L]. (The fifth chapter of [L] deals with the cohomology of p-adic analytic groups; an adequate treatment of this important subject is beyond the scope of this book, and the competence of the present authors. For a recent account of some of Lazard's results the reader is referred to J. S. Wilson's book 'Profinite groups' (Wilson 1998); a fuller treatment is given in the chapter by Symonds and Weigel in [DSS].) Exactly which aspects of the theory of Lie groups over 'nonclassical' local fields are due to Lazard, Serre or Bourbaki, respectively, we do not know; but the central topic of this book, namely how the group-theoretic properties of a pro-p group reflect its status as a p-adic analytic group, is entirely the brainchild of Lazard.

Dependence of chapters