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FUNCTIONAL EQUATIONS AND UNIFORMITY FOR LOCAL ZETA FUNCTIONS OF NILPOTENT GROUPS

By MARCUS P. F. DU SAUTOY and ALEXANDER LUBOTZKY

Abstract. We investigate in this paper the zeta function $\zeta_{\Gamma,p}^{\wedge}(s)$ associated to a nilpotent group Γ introduced in [GSS]. This zeta function counts the subgroups $H \leq \Gamma$ whose profinite completion \hat{H} is isomorphic to the profinite completion $\hat{\Gamma}$. By representing $\zeta_{\Gamma,p}^{\wedge}(s)$ as an integral with respect to the Haar measure on the algebraic automorphism group G of the Lie algebra associated to Γ and by generalizing some recent work of Igusa [I], we give, under some assumptions on Γ , an explicit finite form for $\zeta_{\Gamma,p}^{\wedge}(s)$ in terms of the combinatorial data of the root system of G and information about the weights of various representations of G . As a corollary of this finite form we are able to prove (1) a certain uniformity in p confirming a question raised in [GSS]; and (2) a functional equation that the local factors satisfy $\zeta_{\Gamma,p}^{\wedge}(s)|_{p \rightarrow p^{-1}} = (-1)^n p^{as+b} \zeta_{\Gamma,p}^{\wedge}(s)$. This functional equation is perhaps the most important result of the paper as it is a new feature of the theory of zeta functions of groups.

0. Introduction. Let Γ be a finitely generated, torsion-free nilpotent group and define for a family of subgroups \mathcal{X}^* of Γ the associated Dirichlet series

$$\zeta_{\Gamma}^*(s) = \sum_{H \in \mathcal{X}^*} |\Gamma : H|^{-s} = \sum_{n=1}^{\infty} a_n^*(\Gamma) n^{-s}$$

where

$$a_n^*(\Gamma) = \text{card}\{H \in \mathcal{X}^* \mid |\Gamma : H| = n\}.$$

Such functions were first introduced in the paper [GSS] where the following classes of subgroups were considered:

$$\begin{aligned} \mathcal{X}^{\leq} &= \{\text{all subgroups of finite index in } \Gamma\} \\ \mathcal{X}^{\triangleleft} &= \{H \in \mathcal{X}^{\leq} \mid H \text{ normal in } \Gamma\} \\ \mathcal{X}^{\cong} &= \{H \in \mathcal{X}^{\leq} \mid H \cong \Gamma\} \\ \mathcal{X}^{\wedge} &= \{H \in \mathcal{X}^{\leq} \mid \hat{H} \cong \hat{\Gamma}\} \end{aligned}$$

where $\hat{\Gamma}$ denotes the profinite completion of Γ . In that paper [GSS] it was estab-

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lished that for $* \in \{\leq, \triangleleft, \wedge\}$ these functions have “Euler products”: if we define for each prime p the following local zeta function

$$\zeta_{\Gamma,p}^*(s) = \sum_{n=0}^{\infty} a_{p^n}^*(\Gamma) p^{-ns}$$

then for $* \in \{\leq, \triangleleft, \wedge\}$

$$\zeta_{\Gamma}^*(s) = \prod_{p \text{ prime}} \zeta_{\Gamma,p}^*(s).$$

They also provide an example to show that $\zeta_{\Gamma}^{\approx}(s)$ does not necessarily decompose in such a manner (see Lemma 7.4 [GSS]). The main result of the paper [GSS] is that the local factors $\zeta_{\Gamma,p}^*(s)$ are rational functions in p^{-s} . This rationality result is extended in [duS1] and [duS2] to the classes of p -adic analytic groups and finitely generated groups of finite rank and in [duS3] to a class of S -arithmetic groups. The proofs all depend in part on expressing $\zeta_{\Gamma,p}^*(s)$ as a p -adic integral with respect to the additive Haar measure on \mathbf{Z}_p^N and then using logical techniques introduced by Denef [D] and later extended by Denef and van den Dries [DvdD] to prove rationality results for the class of “definable” p -adic integrals.

In [duS4] the logical setting is exploited further to prove a number of uniformity results. Although powerful, these techniques give us little control on the resulting rational functions. In particular they are insufficient to give us an answer to the following question, raised in [GSS], about the uniformity in p of these rational functions in p^{-s} :

Question 0.1. Let $* \in \{\leq, \triangleleft, \wedge\}$ and Γ be a finitely generated, torsion-free nilpotent group. Do there exist finitely many rational functions $W_1^*(Y, X), \dots, W_r^*(Y, X) \in \mathbf{Q}(Y, X)$ such that for each prime p there exists i such that

$$\zeta_{\Gamma,p}^*(s) = W_i^*(p, p^{-s})?$$

Definition 0.2. If the answer to Question 1.1 is “yes”, we shall say that $\zeta_{\Gamma}^*(s)$ is *almost universal horizontally* or *almost universal in p*. If $r = 1$ we drop the “almost”.

(The qualification “horizontally” in this definition refers to the fact that we are varying the prime p . In [duS5] the concept of being almost universal “vertically” is introduced in which the prime p is fixed and one considers how $\zeta_{\Gamma,p}^*(s)$ varies in a tower of unramified extensions of \mathbf{Q}_p . In the work of Igusa it is this vertical direction which has received more attention. The two concepts seem to be intimately related in the sense that once you can prove one direction, the proof generally yields universality in the other direction. In our context the global

zeta function and its Euler product make the horizontal direction more natural to consider.)

The Question 0.1 arose from experimental observation. For example, if ϑ_K denotes the ring of integers of a number field K and $\Gamma = \text{Tr}_3(\vartheta_K)$ is the group of upper triangular matrices over ϑ_K then the form of $\zeta_{\Gamma,p}^*(s)$ depends exactly on how the prime p behaves in the extension field K . In [GSS] they go on to answer this question affirmatively for the class of finitely generated free nilpotent groups and $* = \wedge$ by producing an explicit formula in this setting for $\zeta_{\Gamma,p}^\wedge(s)$.

In this paper we generalize this example to produce an explicit formula for $\zeta_{\Gamma,p}^\wedge(s)$ for a wide class of finitely generated, torsion-free nilpotent groups. We get two important corollaries of this explicit result. In this setting

- (1) $\zeta_{\Gamma}^\wedge(s)$ is almost universal in p ;
- (2) the rational functions $W_i(Y, X)$ satisfy a functional equation of the following form

$$W_i(Y^{-1}, X^{-1}) = (-1)^{n_i} Y^{a_i} X^{b_i} W_i(Y, X)$$

where n_i, a_i and b_i are explicitly computable integers.

It is perhaps this functional equation which is the most important result of this paper. Although it is only a local functional equation, it is a new feature in the theory of zeta functions of groups. This surprising symmetry lends more weight to the claim that these functions are important natural invariants of a group. In addition all known examples of the other (perhaps more interesting) zeta functions $\zeta_{\Gamma,p}^*(s) * \in \{\leq, \triangleleft\}$ that have been calculated also satisfy such a functional equation. This experimental evidence is documented in [duS5]. Such a functional equation hints perhaps at an analogue of the methods we shall introduce here which would yield more precise knowledge about the functions $\zeta_{\Gamma,p}^*(s) * \in \{\leq, \triangleleft\}$.

The key to our calculation is to represent our zeta function as an integral with respect to the Haar measure on an algebraic group G , the automorphism group of the Lie algebra associated to Γ (see Proposition 1.1). In contrast to the integrals of [GSS], [duS1] and [duS2], we are able to exploit methods which yield an explicit formula in terms of combinatorial data associated to the root system of the algebraic group G and information about the weights of representations of G .

In §1 we explain this integral representation. This integral can in fact be defined for any algebraic group G over a number field k and a representation ρ as follows:

Definition 0.3. Let G be a linear algebraic group defined over a field k and fix a k -rational representation $\rho : G \rightarrow \text{GL}_n$.

- (i) If k is a finite extension of \mathbf{Q}_p we set

$$Z_{G(k), \rho}(s) = \int_{G^+} |\det \rho(g)|^s \mu_G(g)$$

where $G^+ = \rho^{-1}(\rho(G(k)) \cap M_n(\vartheta_k))$, ϑ_k is the ring of integers of k and μ_G denotes the right Haar measure on $G(k)$ normalized such that $\mu_G(G(\vartheta_k)) = 1$.

(ii) If k is a finite extension of \mathbf{Q} and \mathfrak{p} is a prime of k we define

$$Z_{G(k),\rho,\mathfrak{p}}(s) = Z_{G(k_{\mathfrak{p}}),\rho}(s).$$

(When it is clear from the context which field k we are considering, we shall often drop the reference to k and write $Z_{G,\rho}(s)$ and $Z_{G,\rho,\mathfrak{p}}(s)$.)

We also define a global zeta function associated to G by considering the Euler product of these local factors

$$Z_{G(k),\rho}(s) = \prod_{\mathfrak{p}} Z_{G(k),\rho,\mathfrak{p}}(s).$$

We can represent $Z_{G(k),\rho}(s)$ as an integral over the adelic points of G .

This zeta function associated to an algebraic group is not a new zeta function, but has been studied by Hey, Weil, Tamagawa, Satake, Macdonald and more recently by Igusa for various reductive groups (see [He], [I], [M], [T], [W] and references therein). Their interest in this function arose from the fact that it generalizes the Dedekind zeta function of a number field K : let $G = \mathbf{G}_m$, the multiplicative group and ρ the natural representation into GL_1 ; then for each prime \mathfrak{p} of ϑ_K we have $Z_{G,\rho,\mathfrak{p}}(s) = \zeta_{K,\mathfrak{p}}(s)$, the Euler \mathfrak{p} -factor of the Dedekind zeta function $\zeta_K(s)$. In §1 we provide some of the history of this previous work which included calculations for GL_n (see Example 1.4 (1)) and GSp_{2n} (see Example 1.4 (2)).

The subsequent sections (§2-§6) are dedicated to evaluating this integral for certain algebraic groups and can be viewed as a contribution to the existing work on this noncommutative generalization of the Dedekind zeta function. Our work can also be viewed as giving an interpretation to the integral of Definition 0.3 as a generating function counting substructures of algebras. This puts it in line with the classical work on zeta functions which counts ideals in rings of algebraic integers or simple algebras.

In §2 we start with the setting of an algebraic group G defined over a finite extension of the local field \mathbf{Q}_p . We catalogue a number of conditions (Assumptions 2.1, 2.2, 2.3) under which we can replace the zeta function $Z_{G,\rho}(s)$ associated with G by an integral over the connected component of the reductive part of G , involving representations describing the action of the reductive part of G on the unipotent radical. We then have to consider the following generalization of $Z_{G,\rho}(s)$.

Definition 0.4. Let G be a linear algebraic group over a field k . Let $\rho : G \rightarrow \mathrm{GL}_n$ be a k -rational representation, $\beta \in \mathrm{Hom}_k(G, \mathbf{G}_m)$ a k -rational character and $\theta : G \rightarrow \mathbf{R}$ an arbitrary function on G .

(i) If k is a finite extension of \mathbf{Q}_p we define

$$Z_{G(k),\rho,\beta,\theta}(s) = \int_{G^+} |\beta(g)|^s \theta(g) \mu_G(g).$$

(ii) If k is a finite extension of \mathbf{Q} and \mathfrak{p} is a prime of k we define

$$Z_{G(k),\rho,\beta,\theta,\mathfrak{p}}(s) = Z_{G(k_{\mathfrak{p}}),\rho,\beta,\theta}(s).$$

Unfortunately, the θ arising from the reduction of §2 need not in general be a character of G . We give an example in §3.4 to demonstrate this, namely $U_4^0(\mathbf{Q}_p)$, the upper triangular matrix algebra. As yet the only cases we can deal with in §5 are when θ is a character. Nonetheless we are able to calculate $\zeta_{\Gamma,p}^{\wedge}(s)$ in this particular example with the encouraging corollary that the result still satisfies a functional equation.

We also use the reduction of §2 to calculate a number of other explicit examples of $\zeta_{\Gamma,p}^{\wedge}(s)$ for certain nilpotent groups. In §3.1 we begin with the free nilpotent group F of class c on d generators. Its automorphism group modulo the IA-automorphisms is isomorphic to GL_d . (The IA-automorphisms are those automorphisms which act trivially on F/F' where F' denotes the derived group. The group of IA-automorphisms is a unipotent subgroup.) Hence combining §2 with Example 1.8 (1) we can compute $\zeta_{F,p}^{\wedge}(s)$. This example has already been calculated using a different approach in [GSS]. In §3.2 we generalize the example of §3.1 to give an expression for $\zeta_{\Gamma,p}^{\wedge}(s)$ for a nilpotent group free in some variety (e.g. metabelian). In §3.3 we construct examples of rings whose automorphism groups are classical groups modulo the IA-automorphisms. In particular we can use Example 1.8 (2) to evaluate $\zeta_{L,p}^{\wedge}(s)$ explicitly in a number of these cases. We refer to [duS5] for some further explicit computations of $Z_{G,\rho}(s)$ for classical groups.

The explicit examples of §3 are valid for all primes p . We show in §4 that in general, if we start with an algebraic group defined over a global number field K , then for almost all primes \mathfrak{p} we can make the reduction of §2 from $Z_{G,\rho,\mathfrak{p}}(s)$ to a zeta function $Z_{H,\rho,\beta,\theta,\mathfrak{p}}(s)$ associated to the connected component of the reductive part of G . This entails showing that $G(K_{\mathfrak{p}})$ satisfies the Assumptions 2.1, 2.2 and 2.3 for almost all primes \mathfrak{p} .

In §5 we consider the question of evaluating the zeta functions $Z_{H,\rho,\beta,\theta}(s)$ arising from the reduction of §2. We slightly generalize some recent work of Igusa [I] to evaluate this zeta function under certain hypotheses on H (Assumptions 5.1–5.5). To make his calculation Igusa utilizes the p -adic Bruhat decomposition associated to the reductive group $H(K_{\mathfrak{p}})$. The conclusion is an explicit finite form for $Z_{H,\rho,\beta_1,\beta_2}(s)$ in terms of certain combinatorial data associated to the root system of H and information about the weights of the irreducible components of

ρ . In Igusa's calculation, the symmetry between positive and negative roots of H gives rise to a functional equation which this expression satisfies.

In §6 we return to the perspective of §4 of a connected reductive algebraic group over a global field K . However, unlike in §4, we cannot remove all the assumptions needed for the computation of §5 by excluding finitely many primes. Our present work therefore calls for even further generalizations of Igusa's calculation. The conclusion of this present paper is then an explicit finite form for $Z_{G,\rho,\mathfrak{p}}(s)$ for almost all primes \mathfrak{p} , whenever G is an algebraic group, over a global field K , with the following conditions on H , the connected component of the reductive part of G :

Assumption 1. The function $\theta : H \rightarrow \mathbf{R}$ is a character (where θ is defined in §2).

Assumption 2. H has a K -split maximal torus.

Assumption 3. The maximal central torus of H is one-dimensional.

Assumption 4. There exists an irreducible component ρ_1 of ρ which 'dominates' the remaining irreducible components.

(In fact we do slightly better than this, by allowing H to be the restriction of scalars of a group \mathcal{H} over $L \geq K$ which satisfies Assumptions 1-4. This provides some non-split algebraic groups for which the present methods work. The behaviour of $Z_{G,\rho,\mathfrak{p}}(s)$ then depends on how the prime \mathfrak{p} behaves in L .) Assumption 1 refers to the action of H on the unipotent radical of G . Note that all classical groups satisfy Assumptions 2 and 3.

We have the following corollaries of this expression. The first Theorem is a generalization of Igusa's work to some nonreductive groups over a global field and some non-irreducible representations.

THEOREM A. *Let G be an algebraic group over a field K where K is a finite extension of \mathbf{Q} and let ρ be a K -rational representation. Suppose G and ρ satisfy Assumptions 1-4. Then*

- (1) *$Z_{G,\rho}(s)$ is almost universal in \mathfrak{p} .*
- (2) *For almost all primes \mathfrak{p} of K , $Z_{G,\rho,\mathfrak{p}}(s)$ satisfies a functional equation of the form*

$$Z_{G,\rho,\mathfrak{p}}(s)|_{p \rightarrow p^{-1}} = (-1)^n p^{as+b} Z_{G,\rho,\mathfrak{p}}(s)$$

for certain explicitly computable integers n, a and b .

THEOREM B. *Let Γ be a finitely generated torsion-free nilpotent group or a ring additively isomorphic to \mathbf{Z}^d whose algebraic automorphism group satisfies*

Assumptions 1-4. Then

- (1) $\zeta_{\Gamma}^{\wedge}(s)$ is almost universal in p ; and
- (2) for almost all primes p , $\zeta_{\Gamma,p}^{\wedge}(s)$ satisfies a functional equation of the form

$$\zeta_{\Gamma,p}^{\wedge}(s)|_{p \rightarrow p^{-1}} = (-1)^n p^{as+b} \zeta_{\Gamma,p}^{\wedge}(s),$$

for certain explicitly computable integers n, a and b .

Of course, the host of conditions we put on the groups in Theorem A and B limit the examples of nilpotent groups for which these results apply. We give in §3 some such examples including all finitely generated nilpotent groups free in some variety (see §3.2) and some class two nilpotent groups whose automorphism groups involve various classical groups (see §3.2). From these examples we can build more by taking for instance direct products. The hope is, however, that this paper will serve as a first step towards a more general result in which we can remove the conditions in Theorems A and B.

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1. From nilpotent groups to algebraic groups. The following two results already established in [GSS] hold the key to our calculation of $\zeta_{\Gamma,p}^{\wedge}(s)$.

The first result is an integral representation for the following zeta function associated with a ring L , additively isomorphic to \mathbf{Z}^n (or \mathbf{Z}_p^n): let $c_m(L)$ denote the number of subrings H with $H \otimes \mathbf{Z}_p \cong L \otimes \mathbf{Z}_p$ for all primes p of index m in L , and define

$$\zeta_L^{\wedge}(s) = \sum_{m=1}^{\infty} c_m(L) m^{-s}.$$

The function $\zeta_L^{\wedge}(s)$ decomposes as an Euler product of the associated local factors $\zeta_{L,p}^{\wedge}(s)$. Let G be the algebraic group defined over \mathbf{Q} (or \mathbf{Q}_p) such that

$$G(F) = \text{Aut}_F(L \otimes F)$$

for every extension field F of \mathbf{Q} (or \mathbf{Q}_p). The choice of a basis for the lattice L inside $L \otimes \mathbf{Q}$ (or $L \otimes \mathbf{Q}_p$) defines a faithful rational representation

$$\rho : G \rightarrow \text{GL}_n$$

with the property that $G(\mathbf{Z}_p) = \rho^{-1}(\rho(G(\mathbf{Q}_p)) \cap \text{GL}_n(\mathbf{Z}_p)) = \text{Aut}(L \otimes \mathbf{Z}_p)$.

PROPOSITION 1.1. *For each prime p*

$$\zeta_{L,p}^\wedge(s) = \int_{G_p^+} |\det \rho(g)|_p^s \mu_G(g)$$

where $G_p^+ = \rho^{-1}(\rho(G(\mathbf{Q}_p)) \cap M_n(\mathbf{Z}_p))$ and μ_G denotes the right Haar measure on the group $G(\mathbf{Q}_p)$ normalized such that $\mu_G(G(\mathbf{Z}_p)) = 1$ and $|\cdot|_p$ is the p -adic norm on \mathbf{Q}_p .

The proof is straightforward and can be found in §3 of [GSS].

There are certain classes of groups for which one can define an associated Lie algebra L such that for almost all primes p the zeta functions $\zeta_{\Gamma,p}^*(s)$ can be replaced by the zeta functions $\zeta_{L,p}^*(s)$. In §4 of [GSS] this is done for the class of finitely generated torsion-free nilpotent groups. To each such group Γ there is associated a Lie algebra $L_\Gamma(\mathbf{Q})$ over \mathbf{Q} (the Lie algebra corresponding to the Malcev completion $\Gamma^\mathbf{Q}$ of Γ under the Malcev correspondence). The injective map $\log : \Gamma \rightarrow L_\Gamma(\mathbf{Q})$ has the property that the set $\log \Gamma$ spans $L_\Gamma(\mathbf{Q})$. In general, $\log \Gamma$ will not be an additive subgroup of $L_\Gamma(\mathbf{Q})$. However in §4 of [GSS] the following result is established:

PROPOSITION 1.2. *Let Γ be a finitely generated torsion-free nilpotent group of Hirsch length n . Then there exists $f \in \mathbf{N}$, depending only on n , such that $L = \log \Gamma^f$ is a Lie subring of $L_\Gamma(\mathbf{Q})$ and, for $* \in \{\leq, \triangleleft, \wedge\}$ and all primes p not dividing f ,*

$$\zeta_{\Gamma,p}^*(s) = \zeta_{L,p}^*(s).$$

If $\log \Gamma$ is a lattice inside $L_\Gamma(\mathbf{Q})$ then we can do slightly better:

PROPOSITION 1.3. *Let Γ be a finitely generated torsion-free nilpotent group. Suppose that $L = \log \Gamma$ is a lattice inside $L_\Gamma(\mathbf{Q})$. Then for all primes p*

$$\zeta_{\Gamma,p}^\wedge(s) = \zeta_{L,p}^\wedge(s).$$

Proof. We have that $\zeta_{\Gamma,p}^\wedge(s) = \zeta_{\hat{\Gamma}_p}^\wedge(s)$ and $\zeta_{L,p}^\wedge(s) = \zeta_{L \otimes \mathbf{Z}_p}^\wedge(s)$ where $\hat{\Gamma}_p$ is the pro- p completion of Γ . Let $H \leq \hat{\Gamma}_p$ with $H \cong \hat{\Gamma}_p$. Then $\log H$ is a Lie subring of $L \otimes \mathbf{Z}_p$ and $\log H \cong L \otimes \mathbf{Z}_p$. Conversely let $M \leq L \otimes \mathbf{Z}_p$ and $M \cong L \otimes \mathbf{Z}_p$. Then M is closed under the Campbell-Hausdorff operation. Hence the image $\exp M$ in $\hat{\Gamma}_p$ defines a subgroup which is isomorphic to $\hat{\Gamma}_p$ since $M \cong L \otimes \mathbf{Z}_p$. The map \log is index preserving by the proof of Lemma 4.10 [GSS]. Hence the result follows. \square

(Note that in [duS4] we considered another class of groups for which $\zeta_{\Gamma,p}^\wedge(s) = \zeta_{L(\Gamma),p}^\wedge(s)$, namely the class of uniform pro- p groups. This equality depends in part on [II]. However in general the automorphism group of $L(\Gamma)$ will fail to have good reduction mod p —see Assumption (5.3). So our present methods will not give any information about $\zeta_{\Gamma,p}^\wedge(s)$.)

By Propositions 1.1, 1.2 and 1.3, the study of $\zeta_{\Gamma,p}^\wedge(s)$ for Γ a finitely-generated, torsion-free nilpotent group reduces to the problem of evaluating the integral arising in Proposition 1.1.

As we mentioned in the Introduction, this integral can be defined for any algebraic group G over a number field K and a rational representation ρ . Since it represents a generalization of the Dedekind zeta function of a number field it has in fact received a certain amount of previous attention. Before we proceed to our analysis of this integral we review some of this history.

Tamagawa [T] considered the case $G = \mathrm{GL}_n$ with the natural representation and proved that the global zeta function (defined as the Euler product of these local zeta functions) has meromorphic continuation to the whole complex plane and satisfies a functional equation similar to $\zeta_K(s)$. The zeta function attached to GL_n is in fact the zeta function of a simple algebra A over the rational number field \mathbf{Q} defined by Hey (see [De]). Consider the arbitrary maximal order ϑ of A and define

$$\zeta_A(s) = \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}$$

where the summation is taken over all the left integral ideals \mathfrak{a} of ϑ . Then $\zeta_A(s)$ is independent of the choice of the maximal order ϑ . If A is the full matrix algebra of degree n over the field K then $\zeta_A(s) = \prod_{\mathfrak{p}} Z_{\mathrm{GL}_n, \mathfrak{p}}(s)$ where \mathfrak{p} runs over the prime ideals of the maximal order ϑ . Zorn [Z] gave a proof using the zeta function of a simple algebra of the local-global theorem of Hasse-Brauer-Noether that a simple algebra A is a full matrix ring over K if and only if all local algebras $A \otimes K_{\mathfrak{p}}$ are full matrix rings over $K_{\mathfrak{p}}$. In a sense we can view Proposition 1.1 as a generalization of this interpretation of the zeta function of an algebraic group as the zeta function of some algebra. In his investigation of zonal spherical functions, Satake [Sa] began the calculation for $G = \mathrm{GSp}_{2n}$, the general symplectic group, proving that it was a rational function. Subsequently Macdonald [M] completed the calculation giving an explicit finite form for the zeta function. We record here both Tamagawa's result and the first few cases of Satake and Macdonald's calculation.

Examples 1.4. (1) Fix a finite extension K of \mathbf{Q} . Let $G = \mathrm{GL}_n$ and $\rho : G(K) \rightarrow \mathrm{GL}_n(K)$ the natural representation. Then for each prime \mathfrak{p} of K

$$Z_{G,\rho,\mathfrak{p}}(s) = \prod_{i=0}^{n-1} \zeta_{K,\mathfrak{p}}(s-i)$$

where $\zeta_{K,\mathfrak{p}}(s)$ denotes the Euler \mathfrak{p} -factor of the Dedekind zeta function $\zeta_K(s)$. Note that the meromorphic continuation and global functional equation of the Euler product $\prod_{\mathfrak{p}} Z_{G,\rho,\mathfrak{p}}(s)$ follows from the corresponding properties of $\zeta_K(s)$.

(2) Let $G = \mathrm{GSp}_{2n}$, the group of symplectic similitudes defined as follows: let i denote the $n \times n$ matrices with 1's along the reverse diagonal and zeros elsewhere, and let

$$j = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

then $\mathrm{GSp}_{2n}(K) = \{x \in \mathrm{GL}_{2n}(K) | xj^t x = \mu(x)j \text{ for some } \mu(x) \in K^*\}$ where $(^t x)$ denotes the transpose of x . Let $\rho : \mathrm{GSp}_{2n}(K) \rightarrow \mathrm{GL}_{2n}(K)$ be the natural representation. Fix a prime \mathfrak{p} of K and denote by q the residue degree of $K_{\mathfrak{p}}$.

(i) If $n = 1$ then

$$Z_{G,\rho,\mathfrak{p}}(s) = \frac{1}{(1 - q^{-s})(1 - q^{1-s})}.$$

(Since $G(K) \cong \mathrm{GL}_2(K)$, this is of course a special case of (1).)

(ii) If $n = 2$ then

$$Z_{G,\rho,\mathfrak{p}}(s) = \frac{(1 + q^{1-2s})}{(1 - q^{-2s})(1 - q^{2-2s})(1 - q^{3-2s})}.$$

(iii) If $n = 3$ then

$$Z_{G,\rho,\mathfrak{p}}(s) = \frac{(1 + q^{1-3s} + q^{2-3s} + q^{3-3s} + q^{4-3s} + q^{5-6s})}{(1 - q^{-3s})(1 - q^{3-3s})(1 - q^{5-3s})(1 - q^{6-3s})}.$$

The first two examples can be found explicitly in [Sa]. The Euler product of these functions can be expressed in terms of $\zeta_K(s)$ and hence have meromorphic continuation and satisfy a global functional equation.

Example (iii) is based on our own calculation using Macdonald's explicit formula [M]. In [duS5] it is proved that the Euler product in Example (2) (iii) has a natural boundary at $\Re(s) = 4/3$. Thus there is no hope of extending Tamagawa's global results to more general algebraic groups.

Note that in example (1) the zeta function associated to GL_n and $K = \mathbf{Q}$ is none other than the zeta function $\zeta_{\mathbf{Z}^n,p}^{\wedge}(s) = \zeta_{\mathbf{Z}^n,p}^{\leq}(s)$ counting subgroups in the free abelian group \mathbf{Z}^n since $\mathrm{GL}_n(\mathbf{Q})$ is the automorphism group of the trivial Lie algebra \mathbf{Q}^n .

In the next section we begin our analysis of these integrals. Are there any restrictions on the sort of algebraic groups G that can arise in our setting? By a

result of Bryant and Groves [BG], any \mathbf{Q} -algebraic group with any given representation can be realized as the automorphism group of a nilpotent \mathbf{Q} -Lie algebra \mathcal{L} modulo the group of its IA-automorphisms. (The IA-automorphisms are those automorphisms which act trivially on \mathcal{L}/\mathcal{L}' where \mathcal{L}' denotes the derived Lie algebra. The group of IA-automorphisms is a unipotent subgroup.) So for our applications we need to consider $Z_{G,\rho,\mathfrak{p}}(s)$ for any algebraic group G .

2. Reduction to reductive groups. Let G be a linear algebraic group defined over k where k is a finite extension of \mathbf{Q}_p and let ϑ_k be the ring of integers of k . Let $\rho = G \rightarrow \mathrm{GL}_n$ be a faithful k -rational representation. Let $N(k)$ be the unipotent radical of $G(k)$, and let G_0 denote the connected component of G . We can write $G_0(k)$ as a semidirect product of $N(k)$ and its reductive part, i.e., there exists a reductive k -algebraic subgroup H of G_0 such that $G_0(k) = N(k) \rtimes H(k)$. In this section we will show, under a host of conditions on G and the representation ρ , how to replace the integral

$$Z_{G,\rho}(s) = \int_{G^+} |\det \rho(g)|^s \mu_G(g)$$

defined in Definition 0.3 by an integral over the connected reductive part of the group. In §4 when we consider an algebraic group over a global number field K we shall show that for almost all primes \mathfrak{p} of K these conditions on $G(K_{\mathfrak{p}})$ and ρ are true.

We begin by reducing to the connected component of G . We need the following:

Assumption 2.1. $G(\vartheta_k)$ maps onto $G(k)/G_0(k)$.

Under this assumption we prove:

PROPOSITION 2.1. $Z_{G,\rho}(s) = Z_{G_0,\rho}(s)$.

Proof. Let g_1, \dots, g_n be representatives from $G(\vartheta_k)$ for the left cosets of $G_0(k)$ in $G(k)$.

Claim.

$$(a) G^+ = \bigcup_{i=1}^n g_i G_0^+; \quad (b) G(\vartheta_k) = \bigcup_{i=1}^n g_i G_0(\vartheta_k).$$

We prove (a); (b) follows similarly. The inclusion $G^+ \supseteq \bigcup_{i=1}^n g_i G_0^+$ is clear. If $g \in G^+$ then $g = g_i g_0 \in g_i G_0(k)$ for some i . Since $g_i \in G(\vartheta_k)$, $\rho(g_0) = \rho(g_i^{-1} g) \in \mathrm{M}_n(\vartheta_k)$. Hence $g \in g_i G_0^+$ and the claim (a) is established.

The proposition follows immediately from:

$$\begin{aligned}
Z_{G,\rho}(s) &= \int_{\bigcup_{i=1}^n g_i G_0^+} |\det \rho(g)|^s \mu_G(g) \\
&= \sum_{i=1}^n \int_{G_0^+} |\det \rho(g_i g_0)|^s n^{-1} \mu_{G_0}(g_0) \\
&= Z_{G_0,\rho}(s)
\end{aligned}$$

since $|\det \rho(g_i)| = 1$ and $\mu_G = n^{-1} \mu_{G_0}$ by claim (b). \square

For the rest of this section we shall assume that G is connected. We suppose further that the representation $\rho : G(k) \rightarrow \mathrm{GL}_n(k)$ satisfies the following:

Assumption 2.2. There exists a partition $n = r_1 + \dots + r_c$ such that in the underlying vector space $V = k^n$, if we set $U_i = 0 \times \dots \times 0 \times k^{r_i} \times 0 \dots \times 0$ then U_i is an $H(k)$ -stable subspace and $N(k)$ acts trivially on V_i/V_{i+1} where $V_i = U_i \oplus \dots \oplus U_c$, i.e., ρ decomposes into block form such that $\rho|_{H(k)}$ is block diagonal and $\rho|_{N(k)}$ is unitriangular.

By a change of basis we can always realize Assumption 2.2. However, a change of basis can also change G^+ . So it is important at this stage to assume that we have chosen the representation to satisfy Assumption 2.2. In §4 when we work over a global field K , we shall use the fact that a change of basis does not change G^+ for almost all primes \mathfrak{p} and hence we can drop this assumption in that setting.

Denote by ρ_i the induced representation of $H(k)$ acting on U_i (i.e., the diagonal block entries of $\rho|_{H(k)}$) and $s_i = \dim V/V_{i+1}$.

Let N_i be the kernel of the natural map $\psi_i : N \rightarrow \mathrm{Aut}(V/V_{i+1})$. Define the representation $\varphi_i : G(k)/N_i \rightarrow \mathrm{GL}_n(k)$ by $\varphi_i|_{H(k)} = \rho|_{H(k)}$ and

$$\varphi_i(nN_i) = \begin{pmatrix} \psi_i(n) & 0 \\ 0 & \mathrm{Id}|_{V_{i+1}} \end{pmatrix}.$$

Define $(G/N_i)^+$ to be the integral matrices with respect to this representation φ_i , i.e.

$$(G/N_i)^+ = \varphi_i^{-1}(\varphi_i(G(k)/N_i) \cap (\mathrm{M}_n(\vartheta_k))).$$

Assumption 2.3. If $\bar{g} \in (G/N_i)^+$ then there exists $g \in G^+$ such that $gN_i = \bar{g}$.

An element $n \in N_i/N_{i+1} \subseteq \mathrm{Aut}(V/V_{i+2})$ is determined by its action on a basis $u_1, \dots, u_{s_{i+1}}$ for V/V_{i+2} . If $i = 1, \dots, s_i$ $n(u_j) = u_j + z_j$ for some $z_j \in V_{i+1}/V_{i+2}$

and $n(u_j) = u_j$ if $u_j \in U_{i+1}$. The map $n \mapsto (z_1, \dots, z_{s_i})$ defines an embedding of N_i/N_{i+1} as a k -subspace of $(V_{i+1}/V_{i+2})^{s_i}$. Therefore for each $h \in H$ there is a map

$$\tau(h) : N_i/N_{i+1} \rightarrow (V_{i+1}/V_{i+2})^{s_i}$$

defined by restricting $\rho_{i+1}^{\oplus s_i}$ to the subspace N_i/N_{i+1} .

Denote by μ_H the Haar measure on $H(k)$ normalized such that $H(\vartheta_k)$ has measure 1 and $\mu_{N_i/N_{i+1}}$ (respectively μ_N) be the Haar measure on N_i/N_{i+1} (respectively N) normalized such that $N_i/N_{i+1}(\vartheta_k) = \varphi_i^{-1}(\varphi_i(N_i/N_{i+1}) \cap M_n(\vartheta_k))$ (respectively $N(\vartheta_k)$) has measure 1. By Assumption 2.3, we can choose a topological splitting $N = \prod_{i=1}^{c-1} N_i/N_{i+1}$ such that $N(\vartheta_k) = \prod_{i=1}^{c-1} N_i/N_{i+1}(\vartheta_k)$. Since $G(\vartheta_k) = H(\vartheta_k)N(\vartheta_k)$ we have that $\mu_G = \mu_H \cdot \prod_{i=1}^{c-1} \mu_{N_i/N_{i+1}}$. The problem in reducing to the reductive part H of the group G arises from the fact that $G^+ \neq H^+N^+$. For each $i = 1, \dots, c-1$ we need to define the following functions $\theta_i : H \rightarrow \mathbf{R}$

$$\theta_i(h) = \mu_{N_i/N_{i+1}}(\{n_i \in N_i/N_{i+1} \mid n_i \tau_i(h) \in M_{s_i \times r_{i+1}}(\vartheta_k)\}).$$

THEOREM 2.2. *Under Assumptions 2.1, 2.2 and 2.3*

$$Z_{G,\rho}(s) = \int_{H^+} |\det \rho(h)|^s \prod_{i=1}^{c-1} \theta_i(h) \mu_H(h).$$

Proof. Denote by $\chi_X(g)$ the characteristic function on the subset $X \subseteq G$. Then

$$\begin{aligned} Z_{G,\rho}(s) &= \int_{G(k)} \chi_{G^+}(g) |\det \rho(g)|^s \mu_G(g) \\ &= \int_{H(k)} f(h) \mu_H(h) \\ \text{where } f(h) &= \int_N \chi_{G^+}(nh) |\det \rho(nh)|^s \mu_N(n) \end{aligned}$$

(see [N] p. 87). Since $|\det \rho(nh)| = |\det \rho(h)|$, to prove the theorem it suffices to prove that for every $h \in H(k)$

$$(2.1) \quad \int_N \chi_{G^+}(nh) \mu_N(n) = \chi_{H^+}(h) \prod_{i=1}^{c-1} \theta_i(h).$$

We prove this by induction on c . The case $c = 1$ corresponds to the situation in which G is already reductive. Suppose now that (2.1) is true for $l < c$. Let $\mu_{\overline{N}} = \prod_{i=1}^{c-2} \mu_{N_i/N_{i+1}}$ and $\mu_{N_{c-1}}$ be the Haar measures on $\overline{N} = N/N_{c-1} = \psi_{c-1}(N)$

and N_{c-1} . Then $\mu_N = \mu_{\overline{N}} \cdot \mu_{N_{c-1}}$ and

$$\int_N \chi_{G^+}(nh) \mu_N(n) = \int_{\overline{N}} f(\overline{n}) \mu_{\overline{N}}(\overline{n})$$

where

$$f(\overline{n}) = \int_{N_{c-1}} \chi_{G^+}(n_{c-1} \psi_{c-1}^{-1}(\overline{n}) h) \mu_{N_{c-1}}(n_{c-1}),$$

and $\psi_{c-1}^{-1}(\overline{n}) \in N$ is a lifting of $\overline{n} \in \overline{N}$ to N chosen in such a way that if $\overline{nh} \in (G/N_{c-1})^+$ then $\psi_{c-1}^{-1}(\overline{n})h \in G^+$. This choice is possible by our Assumption 2.3. Now $n_{c-1} \psi_{c-1}^{-1}(\overline{n})h \in G^+$ if and only if the following conditions are satisfied: $\overline{nh} \in (G/N_{c-1})^+$ and

$$\begin{aligned} \rho\left(n_{c-1}(\psi_{c-1}^{-1}(\overline{n})h)\right) &= \rho(n_{c-1}) \cdot \rho(\psi_{c-1}^{-1}(\overline{n})h) \\ &= \begin{pmatrix} I_{r_1} & \dots & 0 & n_{c-1}(1) \\ & \ddots & \vdots & \vdots \\ & & I_{r_{c-1}} & n_{c-1}(c-1) \\ & & & I_{r_c} \end{pmatrix} \\ &\times \begin{pmatrix} \rho_1(h) & \dots & * & n(1) \\ & \ddots & \vdots & \vdots \\ & & \rho_{c-1}(h) & n(c-1) \\ & & & \rho_c(h) \end{pmatrix} \in M_n(\vartheta_k) \end{aligned}$$

where $n_{c-1}(i) \in M_{r_i, r_c}(k)$ and $n(i) \in M_{r_i, r_c}(\vartheta_k)$ and

$$\varphi_{c-1}(\overline{nh}) = \begin{pmatrix} \rho_1(h) & \dots & * & 0 \\ & \ddots & \vdots & \vdots \\ & & \rho_{c-1}(h) & 0 \\ & & & \rho_c(h) \end{pmatrix}.$$

So $n_{c-1} \psi_{c-1}^{-1}(\overline{n})h \in G^+$ if and only if $\overline{nh} \in (G/N_{c-1})^+$ and

$$\begin{pmatrix} n(1) \\ \vdots \\ n(c-1) \end{pmatrix} + \begin{pmatrix} n_{c-1}(1) \\ \vdots \\ n_{c-1}(c-1) \end{pmatrix} \rho_c^{\oplus s_{c-1}}(h) \in M_{s_{c-1} \times r_c}(\vartheta_k).$$

Therefore

$$\begin{aligned} f(\overline{n}) &= \chi_{(G/N_{c-1})^+}(\overline{nh}) \\ &\cdot \mu_{N_{c-1}}\left(\{n_{c-1} \in N_{c-1} \leq V_c^{s_{c-1}} \mid n_{c-1} \tau_{c-1}(h) \in M_{s_{c-1} \times r_c}(\vartheta_k)\}\right) \\ &= \chi_{(G/N_{c-1})^+}(\overline{nh}) \cdot \theta_{c-1}(h). \end{aligned}$$

The theorem then follows by applying the induction hypothesis to the group G/N_{c-1} . \square

In §5 and §6 we shall show how to evaluate such integrals over reductive groups in the case where the θ_i are characters on H . Unfortunately this is not always the case. We give an example in §3 of a nilpotent Lie algebra L such that the maps θ_i associated to the automorphism group of L are not characters.

Note that if $\tau_i(h) : N_i/N_{i+1} \rightarrow N_i/N_{i+1} \leq (V_{i+1}/V_{i+2})^{s_i}$ then $\tau : H \rightarrow \text{Aut } N_i/N_{i+1}$ defines a representation of H and $\theta_i(h) = |\det \tau(h)|^{-1}$, a character of H .

There are two classes of nilpotent groups for which the automorphism groups have this property.

THEOREM 2.3. *Suppose that Γ is a finitely generated torsion-free nilpotent group and G is the automorphism group of the associated Lie algebra and H is the connected component of the reductive part of G . Suppose further that either*

- (i) Γ is a class 2 nilpotent group; or
- (ii) $\Gamma_c = F/\gamma_c F$ where F is the free group in some variety and γ_c is the c th term of the lower central series.

Then $\theta_i : H \rightarrow \mathbf{R}$ ($i = 1, \dots, c-1$) is a character of H where θ_i is defined as above.

Proof. Let $\mathcal{L} = L \otimes \mathbf{Q}$ be the \mathbf{Q} -Lie algebra associated to Γ and $\mathcal{L}_i = \gamma_i \mathcal{L}$. Then \mathcal{L}_i is a G -stable subspace and N , the unipotent radical of G , acts trivially on $\mathcal{L}_i/\mathcal{L}_{i+1}$. Let $u_1, \dots, u_{s_{i+1}}$ be a basis for $\mathcal{L}/\mathcal{L}_{i+2}$. Then for any $z_1, \dots, z_{s_1} \in \mathcal{L}_{i+1}/\mathcal{L}_{i+2}$, the linear map defined by

$$\alpha : u_i \rightarrow u_i + z_i \text{ for } j = 1, \dots, s_1$$

$$\alpha : u_i \rightarrow u_i \text{ for } j = s_1 + 1, \dots, s_{i+1}$$

lifts to an automorphism of \mathcal{L} . Hence N_i/N_{i+1} , under our identification, is the full \mathbf{Q} -space $(\mathcal{L}_{i+1}/\mathcal{L}_{i+2})^{s_1}$ and $\tau_i(h) = \rho_i^{\oplus s_1} : (\mathcal{L}_{i+1}/\mathcal{L}_{i+2})^{s_1} \rightarrow (\mathcal{L}_{i+1}/\mathcal{L}_{i+2})^{s_1}$. As we indicated above this implies that the maps $\theta_i : H \rightarrow \mathbf{R}$ are characters of H . \square

3. Examples.

3.1. Free nilpotent groups and Lie algebras. Let \mathcal{L} be the free nilpotent Lie algebra over \mathbf{Q} of class $c \geq 2$ on $d \geq 2$ generators and let K be a finite extension of \mathbf{Q} of degree n . In this section we calculate the zeta function $\zeta_{\vartheta_K L, p}^\wedge(s)$ where L is a \mathbf{Z} -Lie subring of \mathcal{L} with the property that $\mathcal{L} = L \otimes_{\mathbf{Z}} \mathbf{Q}$. To do this we shall use Theorem 2.2 above. We therefore need to know the structure of the

automorphism group $\text{Aut}_{\mathbf{Q}} K\mathcal{L}$ of $K\mathcal{L}$ as a \mathbf{Q} -algebra. For a general Lie algebra such a question may not have such a simple answer, even with knowledge of $\text{Aut}_K K\mathcal{L}$. However, for the free nilpotent Lie algebra $K\mathcal{L}$, Segal [Se] has shown that this automorphism group is relatively easy to determine. Once we have calculated $\zeta_{\vartheta_K L, p}^\wedge(s)$, we can immediately deduce as a corollary of Proposition 1.2 an expression for the zeta function $\zeta_{G, p}^\wedge(s)$ where G is either the free nilpotent group F of class $c \geq 2$ on $d \geq 2$ generators or F^{ϑ_K} (when this makes sense). We explain what this qualification means at the end of this subsection. The calculation of $\zeta_{G, p}^\wedge(s)$ in this setting was already done in [GSS]. The reader who would prefer to ignore the complication arising from extending the field may easily do so. However since this extension was considered in [GSS], the fact that our answer agrees with that of [GSS] does provide us with a useful check for the formula of §2. Also it provides us with examples where the reductive group H over \mathbf{Q} is non-split but is the restriction of scalars of a split group \mathcal{H} over K , i.e. $\mathcal{H}(K) = \text{GL}_d(K)$, for which we can calculate $Z_{H(K), \rho, \mathfrak{p}}(s)$. We shall generalize this argument when we come to §6.

Let u_1, \dots, u_d be free generators for \mathcal{L} . Let $\gamma_i \mathcal{L}$ denote the i th term of the lower central series of \mathcal{L} and define $r_i = \dim_{\mathbf{Q}} \gamma_i \mathcal{L} / \gamma_{i+1} \mathcal{L}$ and $s_i = \dim_{\mathbf{Q}} \mathcal{L} / \gamma_{i+1} \mathcal{L}$. There are formulas given by Witt for these dimensions:

$$r_i = 1/i \sum_{j|i} \mu(j) d^{i/j}$$

where $\mu(j)$ is the Möbius function (see for example [MKS] Theorem 5.11). There exists a sequence of elements z_1, \dots, z_{s_c} called a *Witt basis* for \mathcal{L} with the property that

- (1) for $s_i + 1 \leq l \leq s_{i+1}$, $z_l = [u_{j_1(l)}, \dots, u_{j_{i+1}(l)}]$ ($j_1(l), \dots, j_{i+1}(l) \in \{1, \dots, d\}$) is a homogeneous Lie commutator of length $i+1$ in the free generators u_1, \dots, u_d ; and
- (2) $z_{s_i+1}, \dots, z_{s_c}$ form a linear basis over \mathbf{Z} for the Lie elements of length $\geq i+1$.

See for example [MKS] §5.6.

We choose the lattice L to be the \mathbf{Z} -span of the basis $\{z_1, \dots, z_{s_c}\}$ and define L_{i+1} to be the \mathbf{Z} -span of $\{z_{s_i+1}, \dots, z_{s_{i+1}}\}$. We then have a decomposition

$$L = L_1 \oplus \dots \oplus L_c.$$

Note that if we take any \mathbf{Z} -Lie subring L' of $K\mathcal{L}$ such that $\mathcal{L} = L' \otimes_{\mathbf{Z}} \mathbf{Q}$, then for almost all primes p

$$\zeta_{L', p}^\wedge(s) = \zeta_{\vartheta_K L, p}^\wedge(s).$$

So by choosing the lattice L we are loosing very little. Let $\{t_1, \dots, t_n\}$ be a basis for ϑ_K over \mathbf{Z} then $\{t_i z_j \mid i = 1, \dots, n; j = 1, \dots, s_c\}$ is a \mathbf{Z} -basis for the Lie

ring $\vartheta_K L$. This basis defines a \mathbf{Q} -rational representation $\rho : G \rightarrow \mathrm{GL}_{ns_c}$ (where $G(F) = \mathrm{Aut}_F(K\mathcal{L} \otimes_{\mathbf{Q}} F)$ for every field extension F of \mathbf{Q}) with the property that $G(\mathbf{Z}) = \mathrm{Aut}_{\mathbf{Z}}(\vartheta_K L)$. By Proposition 1.1, for each prime p ,

$$\zeta_{\vartheta_K L, p}^{\wedge}(s) = \int_{G_p^+} |\det \rho(g)|^s \mu_G(g)$$

where $G_p^+ = \rho^{-1}(\rho(G(\mathbf{Q}_p)) \cap M_{ns_c}(\mathbf{Z}_p))$.

Let $R = K \otimes_{\mathbf{Q}} \mathbf{Q}_p$. (Note that R is not a field but is rather the direct product of fields $K_{\mathfrak{p}}$ for primes \mathfrak{p} of ϑ_K dividing p .) Then $G(\mathbf{Q}_p) = \mathrm{Aut}_{\mathbf{Q}_p}(R\mathcal{L})$. Each \mathbf{Q}_p -linear transformation α of $R\mathcal{L}$ is represented by a $c \times c$ matrix (α_{ij}) with $\alpha_{ij} \in \mathrm{Hom}_{\mathbf{Q}_p}(R\mathcal{L}_i, R\mathcal{L}_j)$. We quote now a result from [Se] which determines the structure of $G_p = G(\mathbf{Q}_p)$.

PROPOSITION 3.1. $G_p = \Gamma_p \rtimes \mathrm{Aut}_{\mathbf{Q}_p} R$ where Γ_p consists of all \mathbf{Q}_p -linear transformations $\alpha = (\alpha_{ij})$ of $R\mathcal{L}$ satisfying

$$\begin{aligned} \alpha_{11} &\in \mathrm{Aut}_R R\mathcal{L}_1 = \mathrm{GL}_d(R) \\ \alpha_{1j} &\in \mathrm{Hom}_R(R\mathcal{L}_1, R\mathcal{L}_j) \quad (2 \leq j \leq c-1) \\ \alpha_{1c} &\in \mathrm{Hom}_{\mathbf{Q}_p}(R\mathcal{L}_1, R\mathcal{L}_c) \\ \alpha_{ij} &= \psi_{ij}(\alpha_{11}, \dots, \alpha_{1,j-i+1}) \quad (2 \leq i \leq j \leq c) \\ \alpha_{ij} &= 0 \quad (i > j) \end{aligned}$$

where ψ_{ij} are \mathbf{Q} -polynomial maps depending only on \mathcal{L} . Also if $\alpha_{11} = \mathrm{Id}_{R\mathcal{L}_1}$ and $\alpha_{ij} = 0$ for $2 \leq j \leq i$ then $\alpha_{kk} = \mathrm{Id}_{R\mathcal{L}_k}$ and $\alpha_{kj} = 0$ for $2 \leq k < j \leq i+k$.

The group Γ_p is the connected component of the \mathbf{Q}_p -algebraic group G_p . Note that it is almost the \mathbf{Q}_p -points of the restriction of scalars $\mathcal{R}_{K/\mathbf{Q}} \mathrm{Aut}_K(K\mathcal{L})$. It is slightly larger because in the top right-hand corner we only demand that $\alpha_{1c} \in \mathrm{Hom}_{\mathbf{Q}_p}(R\mathcal{L}_1, R\mathcal{L}_c)$ rather than $\alpha_{1c} \in \mathrm{Hom}_R(R\mathcal{L}_1, R\mathcal{L}_c)$.

Let N_i be the kernel of the map $\Gamma_p \rightarrow \mathrm{Aut}_{\mathbf{Q}_p}(R\mathcal{L}/\gamma_{i+1}(R\mathcal{L}))$ for each $i = 1, \dots, c$. Each N_i can be identified with the subgroup of Γ_p consisting of $\alpha = (\alpha_{ij})$ with $\alpha_{11} = \mathrm{Id}_{R\mathcal{L}_1}$ (which implies $\alpha_{ii} = \mathrm{Id}_{R\mathcal{L}_i}$) and $\alpha_{1j} = 0$ for $2 \leq j \leq i$ (which implies $\alpha_{kj} = 0$ whenever $2 \leq k < j \leq i+k$). The group N_1 is known as the group of IA-automorphisms and in this case coincides with the unipotent radical of Γ_p . If we set H to be the subgroup of Γ_p consisting of all diagonal elements $\alpha = (\alpha_{ij})$ with $\alpha_{ij} = 0$ for all $i \neq j$, then H is the reductive part of Γ_p and is isomorphic with $\mathrm{GL}_d(R)$.

We are now in a position to prove that the three assumptions of the previous section are true in this setting.

LEMMA 3.2. (1) $G_p(\mathbf{Z}_p)$ maps onto G_p/Γ_p .

(2) The representation ρ decomposes into block form such that $\rho|_H$ is block diagonal and $\rho|_{N_1}$ is unitriangular.

(3) If $\bar{g} \in (\Gamma_p/N_i)^+$ then there exists $g \in \Gamma_p^+$ such that $gN_i = \bar{g}$. (Recall that $(\Gamma_p/N_i)^+$ is defined with respect to the representation $\varphi_i : \Gamma_p/N_i \rightarrow \mathrm{GL}_{nsc}(\mathbf{Q}_p)$ defined in §2.)

Proof. (1) $\mathrm{Aut}_{\mathbf{Q}_p} R$ fixes the subring $\vartheta_K \otimes_{\mathbf{Z}} \mathbf{Z}_p$ and hence also the submodule $\vartheta_K L \otimes \mathbf{Z}_p$. So $\mathrm{Aut}_{\mathbf{Q}_p} R \leq G_p(\mathbf{Z}_p)$ and hence $G_p(\mathbf{Z}_p)$ maps onto G_p/Γ_p .

(2) is immediate.

(3) Since $\{z_1, \dots, z_{s_c}\}$ is a \mathbf{Z} -basis for the Lie ring L with the property that each element of the basis is a homogeneous Lie commutator in the generators $\{z_1, \dots, z_d\} = \{u_1, \dots, u_d\}$, if $g \in \Gamma_p$ and the coefficients of $g(t_i u_j)$ (for $i = 1, \dots, n$ and $j = 1, \dots, d$) with respect to the basis $\{t_i z_j | i = 1, \dots, n; j = 1, \dots, s_c\}$ are all integral then also the coefficients of $g(t_i z_j)$ (for $i = 1, \dots, n$ and $j = 1, \dots, s_c$) are integral. Hence the polynomial maps ψ_{ij} of Proposition 3.1 can be defined over \mathbf{Z} . Hence if $g = (\alpha_{ij}) \in \Gamma_p$ then $g \in \Gamma_p^+$ if and only if $\alpha_{1j} \in \mathrm{Hom}_{\mathbf{Z}_p}(\vartheta_K L_1 \otimes \mathbf{Z}_p, \vartheta_K L_j \otimes \mathbf{Z}_p)$ for $1 \leq j \leq c$.

Choose an element $g' = (\alpha'_{ij}) \in \Gamma_p$ with the property that $g'N_i = \bar{g}$. Then $\alpha_{1j} \in \mathrm{Hom}_{\mathbf{Z}_p}(\vartheta_K L_1 \otimes \mathbf{Z}_p, \vartheta_K L_j \otimes \mathbf{Z}_p)$ for $1 \leq j \leq i$. We are required to find $g \in \Gamma_p^+$ such that $g'N_i = gN_i$. Set $g = (\alpha_{ij})$ where

$$\alpha_{ij} = \alpha'_{ij} \text{ for } j \leq i$$

$$\alpha_{ij} = 0 \text{ for } j > i$$

$$\text{and } \alpha_{ij} = \psi_{ij}(\alpha_{11}, \dots, \alpha_{1j-i+1}) \text{ for } 2 \leq i \leq j \leq c.$$

By Proposition 3.1, $g \in \Gamma_p$ and $g'N_i = gN_i$. But now $\alpha_{1j} \in \mathrm{Hom}_{\mathbf{Z}_p}(\vartheta_K L_1 \otimes \mathbf{Z}_p, \vartheta_K L_j \otimes \mathbf{Z}_p)$ for $1 \leq j \leq c$. Hence $g \in \Gamma_p^+$. This proves (3). \square

Lemma 3.2 allows us now to apply Theorem 2.2. Combining this with Theorem 2.3 we have

$$\begin{aligned} \zeta_{\vartheta_K L, p}^\wedge(s) &= \int_{G_p^+} |\det \rho(g)|^s \mu_G(g) \\ &= \int_{H_p^+} |\det \rho(h)|^s \prod_{i=1}^{c-1} |\det \tau_i(h)|^{-1} \mu_H(h) \\ &= \int_{H_p^+} \prod_{i=1}^c |\det \rho_i(h)|^s \prod_{i=1}^{c-1} |\det \tau_i(h)|^{-1} \mu_H(h) \end{aligned}$$

where $\tau_i : H \rightarrow \mathrm{Aut}(N_i/N_{i+1})$ is the representation of H defined by restricting $\rho_{i+1}^{\oplus s_i n}$ to the subspace N_i/N_{i+1} of $(R\mathcal{L}_{i+1})^{s_i n}$.

To complete the calculation we must analyze the representations ρ_i of H acting on $R\mathcal{L}_i$ and the representations τ_i of H . From our description of N_i , for $i < c - 1$ we can identify N_i/N_{i+1} with $\text{Hom}_R(R\mathcal{L}_1, R\mathcal{L}_{i+1}) = M_{r_1 \times r_{i+1}}(R) \leq M_{r_1 n \times r_{i+1} n}(\mathbf{Q}_p)$ and N_{c-1} with $\text{Hom}_{\mathbf{Q}_p}(R\mathcal{L}_1, R\mathcal{L}_c) = M_{r_1 n \times r_c n}(\mathbf{Q}_p)$. Hence

$$|\det \tau_i(h)| = |\det \rho_{i+1}(h)|^{r_1} \text{ for } i < c - 1$$

and $|\det \tau_{c-1}(h)| = |\det \rho_c(h)|^{r_1 n}$.

This leads to the following expression:

$$(3.1) \quad \zeta_{\vartheta_K L, p}^\wedge(s) = \int_{H_p^+} |\det \rho_1(h)|^s \prod_{i=2}^{c-1} |\det \rho_i(h)|^{s-r_1} |\det \rho_c(h)|^{s-r_1 n} \mu_H(h).$$

By Proposition 3.1, we know that $\rho_i(h) = \alpha_{ii} = \psi_{ii}(\alpha_{11})$ and is thus determined by the matrix $\rho_1(h) = \alpha_{11}$. The following lemma expresses the determinant of $\rho_i(h)$ as a function of $\rho_1(h)$. The proof is essentially a reconstruction (and small correction) of the proof of Lemma 7.8 [GSS] in our context.

LEMMA 3.3. $|\det \rho_i(h)| = |\det \rho_1(h)|^{ir_i/d}$.

Proof. Since by Proposition 3.1 H is in fact the restriction of scalars of a group over R we can in fact consider $\rho(h)$ as an element of $\text{GL}_{s_c}(R)$ where the representation is taken with respect to a Witt basis. Without loss of generality we may suppose that $i = c$. Since $R = K \otimes_{\mathbf{Q}} \mathbf{Q}_p$ is a direct product of the fields $K_{\mathfrak{p}}$ for the primes \mathfrak{p} of ϑ_K dividing p , it suffices to consider $h \in \text{Aut}_{\mathbf{Q}_p}(L \otimes_K K_{\mathfrak{p}})$. Let $\vartheta_{\mathfrak{p}}$ denote the ring of integers of $K_{\mathfrak{p}}$, then $\vartheta_{\mathfrak{p}}$ is a principal ideal domain. Contrary to the statement in the proof of Lemma 7.8 [GSS], $\rho_1(h)$ is not necessarily diagonalizable. However we are required to establish an algebraic identity on the entries of the matrix $\rho(h)$ for $h \in H$. Hence it suffices to prove this identity on a $K_{\mathfrak{p}}$ -Zariski dense subset. Since the subset of H for which $\rho_1(h) \in \text{GL}_d(\mathbf{Q})$ is diagonalizable over $\text{GL}_d(\mathbf{Q})$ is $K_{\mathfrak{p}}$ -Zariski dense, we may assume that $\rho_1(h) \in \text{GL}_d(\mathbf{Q})$ is diagonalizable over $\text{GL}_d(\mathbf{Q})$.

We may further suppose that we have chosen the \mathbf{Q} -basis $\{u_1, \dots, u_d\}$ for $L_1 \otimes \mathbf{Q}$ with the property that $\rho_1(h)$ is a diagonal matrix with respect to this basis since a change of basis does not affect the calculation of the determinant. We extend this basis to a Witt basis $\{z_1, \dots, z_{s_c}\}$ for $L \otimes \mathbf{Q}$. Then by property (1) of a Witt basis $\rho_c(h)$ is also a diagonal matrix with entries $\lambda_{j_1(l)} \cdots \lambda_{j_c(l)}$ for $s_{c-1} + 1 \leq l \leq s_c$ where $\rho_1(h)u_j = \lambda_j u_j$.

Setting $a_j = \text{ord}_{\mathfrak{p}} \lambda_j$ and $|\vartheta_{\mathfrak{p}} : \mathfrak{p}| = q$ we have

$$|\det \rho_c(h)| = \prod_{l=s_{c-1}+1}^{s_c} q^{a_{j_1(l)} + \cdots + a_{j_c(l)}} = q^{f(a_1, \dots, a_d)}.$$

Note however that for any permutation σ of $\{1, \dots, d\}$ we could have chosen a different ordering $u'_1 = u_{\sigma(1)}, \dots, u'_d = u_{\sigma(d)}$ of the free generating set resulting in a different basis $\{[u'_{j_1}, \dots, u'_{j_c}]\}$ for $L_c \otimes \mathbf{Q}$. With respect to this basis

$$|\det \rho_c(h)| = q^{f(a_{\sigma(1)}, \dots, a_{\sigma(d)})}.$$

Hence $f(a_{\sigma(1)}, \dots, a_{\sigma(d)}) = f(a_1, \dots, a_d)$ for any permutation σ . Thus $f(a_1, \dots, a_d)$ is a sum of cm_c terms a_j in which each of a_1, \dots, a_d occurs equally often, and so

$$f(a_1, \dots, a_d) = cm_c(a_1 + \dots + a_d)/d.$$

The result follows. \square

Combining this lemma with (3.1) we have:

COROLLARY 3.4.

$$\zeta_{\vartheta_K L, p}^\wedge(s) = \int_{H_p^+} |\det \rho_1(h)|^{as-b} \mu_H(h)$$

where $a = (r_1 + 2r_2 + \dots + cr_c)/d$ and $b = (2r_2 + \dots + (c-1)r_{c-1}) + ncr_c$.

We now turn to the question of evaluating this integral.

As we pointed out in the proof of Lemma 3.2 (3), $h \in H_p^+$ if and only if

$$\rho_1(h) = \alpha_{11} \in \text{Aut}_R(R\mathcal{L}_1) \cap \text{Hom}_{\mathbf{Z}_p}(\vartheta_K L_1 \otimes \mathbf{Z}_p, \vartheta_K L_1 \otimes \mathbf{Z}_p).$$

Since $K \otimes \mathbf{Q}_p$ (respectively $\vartheta_K \otimes \mathbf{Z}_p$) is a direct product of $K_{\mathfrak{p}}$ (respectively $\vartheta_{K_{\mathfrak{p}}}$) for primes \mathfrak{p} of ϑ_K dividing p , by choosing a basis $t_1, \dots, t_{n_{\mathfrak{p}}}$ for $\vartheta_{K_{\mathfrak{p}}}$ over \mathbf{Z}_p where $[K_{\mathfrak{p}} : \mathbf{Q}_p] = n_{\mathfrak{p}}$, we can write

$$\begin{aligned} \text{Aut}_R(R\mathcal{L}_1) \cap \text{End}_{\mathbf{Z}_p}(\vartheta_K L_1 \otimes \mathbf{Z}_p) &= \prod_{\mathfrak{p} \mid p} \text{Aut}_{K_{\mathfrak{p}}}(K_{\mathfrak{p}} \mathcal{L}_1) \cap \text{End}_{\mathbf{Z}_p}(\vartheta_{K_{\mathfrak{p}}} L_1 \otimes \mathbf{Z}_p) \\ &= \prod_{\mathfrak{p} \mid p} \text{GL}_d(K_{\mathfrak{p}}) \cap \text{M}_{dn_{\mathfrak{p}}}(\mathbf{Z}_p) \\ &= \prod_{\mathfrak{p} \mid p} \text{GL}_d(K_{\mathfrak{p}})^+. \end{aligned}$$

If $\alpha_{11} = (h_{\mathfrak{p}}) \in \text{Aut}_R(R\mathcal{L}_1) = \prod_{\mathfrak{p} \mid p} \text{Aut}_{K_{\mathfrak{p}}}(K_{\mathfrak{p}} \mathcal{L}_1)$ then $\det(\alpha_{11}) = \prod_{\mathfrak{p} \mid p} \det(h_{\mathfrak{p}})$.

Piecing all this information together we can write

$$\zeta_{\vartheta_K L, p}^\wedge(s) = \prod_{\mathfrak{p} \mid p} \int_{\mathrm{GL}_d(K_{\mathfrak{p}})^+} |\det(h_{\mathfrak{p}})|^{as-b} \mu_{\mathrm{GL}_d(K_{\mathfrak{p}})}(h_{\mathfrak{p}})$$

(since $\mu_H = \mu_{\mathrm{GL}_d(R)} = \prod_{\mathfrak{p} \mid p} \mu_{\mathrm{GL}_d(K_{\mathfrak{p}})}$ where $\mu_{\mathrm{GL}_d(K_{\mathfrak{p}})}$ is normalized such that $\mathrm{GL}_d(\vartheta_{K_{\mathfrak{p}}})$ has measure 1).

However, we have already given an expression for the integrals on the right of this equality in Example 1.4 (1). Hence to summarize we have:

THEOREM 3.5. *Let \mathcal{L} be the free nilpotent Lie algebra over \mathbf{Q} of class $c \geq 2$ on $d \geq 2$ generators and let K be a finite extension of \mathbf{Q} of degree n . Let L be the \mathbf{Z} -Lie subring of \mathcal{L} spanned by a Witt basis for \mathcal{L} . Then for each prime p*

$$\zeta_{\vartheta_K L, p}^\wedge(s) = \prod_{\mathfrak{p} \mid p} \prod_{i=0}^{d-1} \zeta_{K, \mathfrak{p}}(as - b - i)$$

where $a = (r_1 + 2r_2 + \cdots + cr_c)/d$ and $b = (2r_2 + \cdots + (c-1)r_{c-1}) + ncr_c$, $r_i = \dim_{\mathbf{Q}} \gamma_i \mathcal{L} / \gamma_{i+1} \mathcal{L}$ and $\zeta_{K, \mathfrak{p}}(s)$ is the Euler \mathfrak{p} -factor of the Dedekind zeta function $\zeta_K(s)$.

Let F denote the free nilpotent group of class $c \geq 2$ on $d \geq 2$ generators. As explained in section 7 of [GSS], if $c = 2$ then F^{ϑ_K} is a torsion-free finitely generated nilpotent group which we shall call G . But in general, F^{ϑ_K} is not a group. However $F^{\vartheta_K \otimes \mathbf{Z}_p}$ is a pro- p group for all $p > c$, and for such p we write $F^{\vartheta_K \otimes \mathbf{Z}_p} = \hat{G}_p$ and define

$$\zeta_{G, p}^\wedge(s) = \zeta_{\hat{G}_p}^\wedge(s)$$

abusing notation as in [GSS]. As we explained in §1, to each torsion-free finitely generated group there is associated a Lie algebra over \mathbf{Q} . In the present setting the Lie algebra associated with F is just the free nilpotent Lie algebra \mathcal{L} over \mathbf{Q} . The injective map $\log : F \rightarrow \mathcal{L}$ maps F^f onto a \mathbf{Z} -Lie subring L of \mathcal{L} for some $f \in \mathbf{N}$. For those p for which \hat{G}_p is defined $\log \hat{G}_p^f = \vartheta_K L \otimes \mathbf{Z}_p \leq R\mathcal{L}$. In Proposition 1.2 we saw that for almost all primes p

$$\zeta_{\hat{G}_p}^\wedge(s) = \zeta_{\vartheta_K L \otimes \mathbf{Z}_p}^\wedge(s).$$

Therefore Theorem 3.5 has the following corollary:

COROLLARY 3.6. *For almost all primes p*

$$\zeta_{G,p}^\wedge(s) = \prod_{\mathfrak{p} \mid p} \prod_{i=0}^{d-1} \zeta_{K,\mathfrak{p}}(as - b - i)$$

where $a = (r_1 + 2r_2 + \dots + cr_c)/d$ and $b = (2r_2 + \dots + (c-1)r_{c-1}) + ncr_c$ and r_i is the rank of $\gamma_i(F)/\gamma_{i+1}(F)$.

Note that in fact by working directly in the group G one can get the result of Corollary 3.6 for all p if $c = 2$ or otherwise for $p > c$ (see [GSS]).

We can also consider the graded Lie ring $\text{gr}(L) = \bigoplus_{i=1}^c L_i/L_{i+1}$. The automorphism group of $\text{gr}(L) \otimes R$ has the same structure as $\text{Aut}_{\mathbb{Q}_p}(RL)$ described in Proposition 3.1 (with different polynomial maps ψ_{ij}). Hence we can perform the same calculation for $\text{gr}(L)$ to calculate $\zeta_{\text{gr}(L),p}^\wedge(s)$.

Definition 3.7. We shall call two Lie rings L_1 and L_2 *isospectral* if they are non-isomorphic but

$$\zeta_{L_1,p}^\wedge(s) = \zeta_{L_2,p}^\wedge(s).$$

We then have the following:

THEOREM 3.8. *If $\text{gr}(L) = \bigoplus_{i=1}^c L_i/L_{i+1}$ denotes the graded Lie ring of the Lie ring L of Theorem 3.5 then $\text{gr}(L)$ and L are isospectral.*

3.2. Nilpotent groups and Lie algebras free in some variety. We have in fact done enough work in the previous section to write down a general formula for a nilpotent Lie algebra free in some variety. Let \mathcal{F} be the free \mathbb{Q} -Lie algebra in some variety and let $\mathcal{F}_c = \mathcal{F}/\gamma_c \mathcal{F}$, a nilpotent Lie algebra of class $c \geq 2$ on d generators, say. Then \mathcal{F}_c is the surjective image of \mathcal{L} , the free nilpotent Lie algebra over \mathbb{Q} of class $c \geq 2$ on d generators considered in the previous section.

Let u_1, \dots, u_d be generators for $\mathcal{F}_c/\gamma_2 \mathcal{F}_c$ and define $r_i = \dim_{\mathbb{Q}} \gamma_i \mathcal{F}_c / \gamma_{i+1} \mathcal{F}_c$ and $s_i = \dim_{\mathbb{Q}} \mathcal{F}_c / \gamma_{i+1} \mathcal{F}$. Then we have the concept of a *Witt basis* z_1, \dots, z_{s_c} of \mathcal{F}_c as defined in §3.1. (To construct such a basis we can take the image of a Witt basis in \mathcal{L} and choose a subset which is a basis for \mathcal{F}_c as a \mathbb{Q} -vector space.)

We again take the lattice L to be the \mathbb{Z} -span of the basis z_1, \dots, z_{s_c} . Note that if we had started with a \mathbb{Z} -Lie algebra L' free in some variety (perhaps with torsion) then, for almost all primes p , $\zeta_{L'/\gamma_c L',p}^\wedge(s) = \zeta_{L,p}^\wedge(s)$ for some such lattice L inside \mathcal{F}_c .

Let $G = \text{Aut}_{\mathbb{Q}}(\mathcal{F}_c)$, then this choice of lattice defines a \mathbb{Q} -rational representation $\rho : G \rightarrow \text{GL}_{s_c}$. It is not hard to verify that the freeness of \mathcal{F}_c implies that the structure of $G(\mathbb{Q}_p)$ is the same as that described in Proposition 3.1. The choice of our lattice as the \mathbb{Z} -span of a Witt basis implies that, as in Lemma 3.2, Assump-

tions 2.1–2.3 are true and hence we can apply Theorem 2.2 and Theorem 2.3 to calculate $\zeta_{L,p}^\wedge(s)$:

$$\begin{aligned}\zeta_{L,p}^\wedge(s) &= \int_{G_p^+} |\det \rho(g)|^s \mu_G(g) \\ &= \int_{H_p^+} \prod_{i=1}^c |\det \rho_i(h)|^s \prod_{i=1}^{c-1} |\det \tau_i(h)|^{-1} \mu_H(h) \\ &= \int_{H_p^+} |\det \rho_1(h)|^s \prod_{i=2}^c |\det \rho_i(h)|^{s-r_1} \mu_H(h).\end{aligned}$$

PROPOSITION 3.9. (1) $|\det \rho_i(h)| = |\det \rho_1(h)|^{ir_i/d}$.

(2) $h \in H_p^+$ if and only if $\rho_1(h) \in M_{r_1}(\mathbf{Z}_p)$.

Proof. (1) The proof of Lemma 3.3 carries through in our setting since it depends only on property (1) of the Witt basis and the fact that the freeness of \mathcal{F}_c implies that any permutation of the basis u_1, \dots, u_d induces an automorphism of L .

(2) This follows from the fact that the polynomials ψ_{ij} defined in Proposition 3.1 will also be defined over \mathbf{Z} in our setting. \square

Since $H_p^+ = (\mathrm{GL}_d)_p^+$ we have exactly the same calculation as above:

THEOREM 3.10. *Let \mathcal{F} be the free \mathbf{Q} -Lie algebra in some variety and let $\mathcal{F}_c = \mathcal{F}/\gamma_c \mathcal{F}$. Let L be \mathbf{Z} -Lie subring of \mathcal{F}_c spanned by a Witt basis for \mathcal{F}_c . Then for each prime p*

$$\zeta_{L,p}^\wedge(s) = \prod_{i=0}^{d-1} \zeta_p(as - b - i)$$

where $a = (r_1 + 2r_2 + \dots + cr_c)/d$ and $b = (2r_2 + \dots + cr_c)$, $r_i = \dim_{\mathbf{Q}} \gamma_i \mathcal{L} / \gamma_{i+1} \mathcal{L}$ and $\zeta_p(s) = 1/(1 - p^{-s})$.

We preferred in the above to ignore the question of calculating $\zeta_{\vartheta_K L, p}^\wedge(s)$ for some field extension K of \mathbf{Q} . If the variety is such that \mathcal{F}_c satisfies the rigidity condition of §2 of [Se] then $\mathrm{Aut}_{\mathbf{Q}}(K\mathcal{F}_c)$ has the same structure as in Proposition 3.1, i.e., almost the restriction of scalars $\mathcal{R}_{K/\mathbf{Q}} \mathrm{Aut}_K(K\mathcal{F}_c)$ except for the top right-hand corner. In this case the above theorem can be refined to read exactly as in Theorem 3.5.

In the previous section we gave formulas for r_i for the free nilpotent groups. Bachmuth considered the variety of metabelian Lie algebras giving corresponding

formulas for r_i :

$$r_i = d(i-1) \binom{i+d-2}{d-2}$$

(see Lemma 3 of [Ba]).

Given these explicit expressions, is it possible to find a free nilpotent \mathbf{Z} -Lie algebra L_1 of class c_1 and a free nilpotent-metabelian \mathbf{Z} -Lie algebra L_2 of class c_2 which are isospectral (in the sense of Definition 3.7)? Despite some concerted effort with the aid of a computer we were unable to find such an example.

As in §3.1 if we take a torsion-free finitely generated group F that is free in some variety then the Lie algebra \mathcal{L} over \mathbf{Q} associated to $F_c = F/\gamma_{c+1}F$ will be free in the corresponding variety. To prove this recall that the Lie algebra corresponds under the Malcev correspondence to the the Malcev completion $F_c^\mathbf{Q}$ of F_c . We can then use the Malcev correspondence to check that the universal property is satisfied for the Lie algebra \mathcal{L} . The injective map $\log : F_c \rightarrow \mathcal{L}$ then maps $F_c^\mathbf{Q}$ onto a \mathbf{Z} -Lie subring L of \mathcal{L} . By Proposition 1.2 we can then deduce:

THEOREM 3.11. *Let $F_c = F/\gamma_{c+1}F$ where F is a torsion-free finitely generated group free in some variety. Then for almost all primes p*

$$\zeta_{F_c,p}^\wedge(s) = \prod_{i=0}^{d-1} \zeta_p(as - b - i)$$

where $a = (r_1 + 2r_2 + \dots + cr_c)/d$, $b = (2r_2 + \dots + cr_c)$ and r_i is the rank of $\gamma_i(F_c)/\gamma_{i+1}(F_c)$.

3.3. Realizing classical groups. In this section we construct examples of \mathbf{Z}_p -rings whose automorphism groups modulo their unipotent radicals are classical groups.

Let V be a vector space of dimension n over \mathbf{Q}_p . We assume that there is defined on V a nonsingular bilinear scalar product $\phi : V \times V \rightarrow \mathbf{Q}_p$. We can use this scalar product to define a \mathbf{Q}_p -algebra structure on $\mathcal{L} = V \times \mathbf{Q}_p$. For (x_1, z_1) and $(x_2, z_2) \in \mathcal{L}$ define

$$(x_1, z_1) * (x_2, z_2) = (0, \phi(x_1, x_2)).$$

This makes \mathcal{L} into a \mathbf{Q}_p -algebra with center $Z(\mathcal{L}) = 0 \times \mathbf{Q}_p$.

Choose a basis $\{x_1, \dots, x_n\}$ for V and define z to be an element of the center $Z(\mathcal{L})$ of \mathcal{L} with the property that $\phi(x_i, x_j) \in \mathbf{Z}_p z$ for all $i, j = 1, \dots, n$. Then the \mathbf{Z}_p -span of $\{x_1, \dots, x_n, z\}$ is a subring of \mathcal{L} which we shall call L . We shall calculate $\zeta_{L,p}^\wedge(s)$. To do this we must know the structure of $G_p = \text{Aut}_{\mathbf{Q}_p}(\mathcal{L})$. This is provided by the following lemma. We denote by $\text{GO}(\phi)$ the nonsingular linear

transformations T of V which are similitudes, i.e., which satisfy the condition $\phi(Tx_1, Tx_2) = \mu(T)\phi(x_1, x_2)$ for all $x_1, x_2 \in V$ where $\mu(T) \in \mathbf{Q}_p^*$.

LEMMA 3.12. G_p is the group consisting of all \mathbf{Q}_p -linear transformations

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbf{Q}_p)$$

satisfying

$$\begin{aligned} \alpha_{11} &\in \mathrm{GO}(\phi) \\ \alpha_{22} &= \mu(\alpha_{11}) \in \mathbf{Q}_p^* \\ \alpha_{12} &\in \mathrm{Hom}_{\mathbf{Q}_p}(V, \mathbf{Q}_p) \cong \mathbf{Q}_p^n. \end{aligned}$$

Proof. Suppose that α is an automorphism of \mathcal{L} . Then $\alpha(x_i) = \alpha_{11}(x_i) + \lambda_i z$ where $\alpha_{11} \in \mathrm{GL}_n(\mathbf{Q}_p)$ and $\alpha(z) = \alpha_{22}z$ since α preserves the centre $Z(\mathcal{L})$. Then

$$\alpha(x_i) * \alpha(x_j) = \phi(\alpha_{11}(x_i), \alpha_{11}(x_j)).$$

Since α is an automorphism we have

$$\alpha(x_i) * \alpha(x_j) = \alpha(x_i * x_j) = \alpha_{22}\phi(x_i, x_j).$$

Hence $\alpha_{11} \in \mathrm{GO}(\phi)$ and $\alpha_{22} = \mu(\alpha_{11})$.

That such a map α defines an automorphism of \mathcal{L} is an easy exercise to check. \square

The reductive part H_p of the group G_p is isomorphic then to the classical group $\mathrm{GO}(\phi)$ and the unipotent radical N_1 is an abelian group isomorphic to \mathbf{Q}_p^n . For this algebraic group (and its representation) the three assumptions 2.1–2.3 are satisfied:

LEMMA 3.13. (1) G_p is connected.
 (2) The representation $\rho : G_p \rightarrow \mathrm{GL}_{n+1}(\mathbf{Q}_p)$ with respect to the basis $\{x_1, \dots, x_n, z\}$ decomposes into block form such that $\rho|_{H_p}$ is block diagonal and $\rho|_{N_1(\mathbf{Q}_p)}$ is unitriangular.
 (3) If $\bar{g} \in (G_p/N_1)^+$ then there exists $g \in G_p^+$ such that $gN_1 = \bar{g}$.

Proof. (1) is well-known and (2) is immediate. (3) Choose an element $g' = (\alpha'_{ij}) \in G_p$ with the property that $g'N_1 = \bar{g}$. If we set $g = (\alpha_{ij})$ where $\alpha_{ii} = \alpha'_{ii}$ for $i = 1, 2$ and $\alpha_{12} = 0$ then $g \in G_p^+$ by Lemma 3.12 and $g'N_1 = gN_1$. \square

We can therefore apply Theorem 2.2 to evaluate $\zeta_{L,p}^\wedge(s)$. Since N_1 is identified with \mathbf{Q}_p^n the map $\theta(h)$ of Theorem 2.2 is just the character $|\det \rho_1(h)|^{-n}$. Hence

$$\zeta_{L,p}^\wedge(s) = \int_{H_p^+} |\det \rho_1(h)|^{s-n} |\det \rho_2(h)|^s \mu_{H_p}(h)$$

where $\rho_1 : H_p \rightarrow \mathrm{GO}(f) \leq \mathrm{GL}_n(\mathbf{Q}_p)$ is the natural representation with respect to the basis $\{x_1, \dots, x_n\}$ and $\rho_2 : H_p \rightarrow \mathbf{Q}_p$ is the representation $\rho_2(h) = \mu(T)$ where $T = \rho_1(h)$. Let A be the matrix representing our bilinear form, i.e. $a_{ij} = \phi(x_i, x_j)$. Then

$$TA({}^t T) = \mu(T)A$$

where $({}^t T)$ denotes the transpose of T . Taking determinants of both sides we see that

$$|\det \rho_2(h)| = |\mu(T)| = |\det T|^{2/n} = |\det \rho_1(h)|^{2/n}.$$

Now if $\rho_1(h) \in \mathrm{M}_n(\mathbf{Z}_p)$ then $\mu(\rho_1(h)) \in \mathbf{Z}_p$ and hence $\rho_2(h) \in \mathbf{Z}_p$. Thus the integral points are determined by ρ_1 and we can write

LEMMA 3.14.

$$\zeta_{L,p}^\wedge(s) = \int_{\mathrm{GO}(\phi)^+} |\det h|^{s(1+2/n)-n} \mu_{\mathrm{GO}(\phi)}(h).$$

This example realizes concretely the zeta functions associated to classical groups as the zeta functions of some \mathbf{Z}_p -ring L .

If the form is skew symmetric then the corresponding ring has the structure of a \mathbf{Z}_p -Lie ring. Suppose that $A = (a_{ij})$ defines such a skew symmetric form. Define a group by the following presentation:

$$G = \langle x_1, \dots, x_n, z \mid [x_i, x_j] = z^{a_{ij}}, [x_i, z] = 1 \rangle.$$

G is a class two nilpotent group. The Lie ring associated then to the pro- p completion of G is precisely the ring constructed above corresponding to the skew symmetric form defined by A . In this manner we can realize the zeta functions of classical groups defined by skew symmetric forms as zeta functions of nilpotent groups (with a change of variable).

For example, if we take

$$G = \langle x_1, x_2, x_3, y_1, y_2, y_3, z \mid [x_i, y_j] = z^{\delta_{ij}}, [y_i, y_j] = [x_i, x_j] = [x_i, z] = [y_i, z] = 1 \rangle$$

then using Lemma 3.14 and Example 1.4 (2) (iii) we get

$$\zeta_{G,p}^\wedge(s) = \frac{(1 + p^{19-4s} + p^{20-4s} + p^{21-4s} + p^{22-4s} + p^{41-8s})}{(1 - p^{18-4s})(1 - p^{20-4s})(1 - p^{22-4s})(1 - p^{23-4s})}.$$

It is proved in [duS5] that the global zeta function $\zeta_G^\wedge(s) = \prod \zeta_{G,p}^\wedge(s)$ cannot be meromorphically continued to the whole complex plane but has a natural boundary.

In §5 we give a formula for $Z_{G,\rho}(s)$ for a \mathbf{Q}_p -split reductive group G involving the combinatorial data associated to G . In the case where the group of similitudes of ϕ splits in \mathbf{Q}_p we can feed in the corresponding data for $G = \mathrm{GO}(\phi)$ to get similar explicit expressions to Examples 1.4. We record some of these examples (calculated with the aid of a computer) in the forthcoming survey [duS5]. Notice that if we had started with a form defined over \mathbf{Q} then $\mathrm{GO}(\phi)(\mathbf{Q}_p)$ may have a different structure depending on the prime p (e.g., the group may be \mathbf{Q}_p -split for some primes but not for others). This behavior will be consistent with a positive answer to Question 0.1 about the uniformity in p of $\zeta_{L,p}^\wedge(s)$ and will be considered in the sequel to this paper when we consider non-split reductive groups ([duS6]).

3.4. The Lie algebra of upper triangular matrices. We give in this section an example of a Lie algebra L with the property that the maps $\theta_i : H \rightarrow \mathbf{R}$ of Theorem 2.2 which we associated to the automorphism group of L are not characters of the group H . Nonetheless it is still possible to calculate the resulting integral with the encouraging corollary that the result still satisfies the functional equation detailed in the Introduction.

Let $\mathrm{U}_n^0(\mathbf{Q}_p)$ denote the Lie algebra of all nilpotent upper triangular $n \times n$ matrices over \mathbf{Q}_p and set $\mathcal{L} = \mathrm{U}_n^0(\mathbf{Q}_p)/\gamma_{c+1}\mathrm{U}_n^0(\mathbf{Q}_p)$ for some $3 \leq c+1 \leq n$. In [Se] Segal gives a description of $\mathrm{Aut}_{\mathbf{Q}_p}\mathcal{L}/N_1$ where N_1 denotes the group of IA-automorphisms. We also need knowledge of the structure of N_1 . It is possible to give a general description of N_1 but here we content ourselves with a specific example.

We consider the algebra $\mathcal{L} = \mathrm{U}_4^0(\mathbf{Q}_p)$ of class 3. Let $G_p = \mathrm{Aut}_{\mathbf{Q}_p}\mathcal{L}$. In fact for this example the description of G_p/N_1 in [Se] is incorrect. We choose a basis for \mathcal{L} given by the standard unit matrices e_{ij} for $1 \leq i < j \leq 4$. Then $\{u_i = e_{ii+1} \mid i = 1, 2, 3\}$ is a set of generators for \mathcal{L} and $\{e_{jj+i} \mid 1 \leq j \leq 4-i\}$ is a basis for the i th layer $\gamma_i\mathcal{L}/\gamma_{i+1}\mathcal{L}$. With respect to this basis we can represent G_p in upper triangular block form (α_{ij}) $1 \leq i, j \leq 3$ where the i th row of the block matrix is the image of $\{e_{jj+i} \mid 1 \leq j \leq 4-i\}$ under the action of G_p . Let $\mathrm{D}_m(\mathbf{Q}_p)$ denote the diagonal subgroup of $\mathrm{GL}_m(\mathbf{Q}_p)$ and let J_m denote the “anti-diagonal” $m \times m$ matrix as in [Se].

PROPOSITION 3.15. G_p consists of all \mathbf{Q}_p -linear transformations $\alpha = (\alpha_{ij}) \in \mathrm{GL}_6(\mathbf{Q}_p)$ $1 \leq i, j \leq 3$ satisfying

$$\begin{aligned}\alpha_{11} &= \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} \cdot J_3^\epsilon \in \mathrm{D}_3(\mathbf{Q}_p) \rtimes \langle J_3 \rangle \\ \alpha_{12} &= \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \\ \lambda_{31} & -\lambda_{11} \end{pmatrix} \in \mathrm{M}_{3,2}(\mathbf{Q}_p) \\ \alpha_{13} &= \begin{pmatrix} * \\ * \\ * \end{pmatrix} \in \mathrm{M}_{3,1}(\mathbf{Q}_p) \\ \alpha_{22} &= \begin{pmatrix} \mu_1\mu_2 & 0 \\ 0 & \mu_2\mu_3 \end{pmatrix} \cdot (-J_2)^\epsilon \in \mathrm{D}_2(\mathbf{Q}_p) \rtimes \langle J_2 \rangle \\ \alpha_{23} &= \psi(\alpha_{11}, \alpha_{12}) \in \mathrm{M}_{2,1}(\mathbf{Q}_p) \\ \alpha_{33} &= \mu_1\mu_2\mu_3 \in \mathbf{Q}_p^* \\ \alpha_{ij} &= 0 \text{ for } (i > j)\end{aligned}$$

where $\epsilon \in \{1, -1\}$ and ψ is a \mathbf{Z} -polynomial map.

Proof. To prove this we consider the sets

$$w^\perp = \{\bar{x} \in \mathcal{L}/\gamma_2(\mathcal{L}) \mid [w, x] \in \gamma_3(\mathcal{L})\}$$

for some $w \equiv a_1u_1 + a_2u_2 + a_3u_3 \pmod{\gamma_2(\mathcal{L})}$. Then

$$\begin{aligned}\dim w^\perp &= 3 \text{ if } w \equiv 0 \\ &= 2 \text{ if } w \not\equiv 0 \text{ and } a_2 = 0 \\ &= 1 \text{ if } a_2 \neq 0.\end{aligned}$$

(Note that in the proof of Proposition 5 of [Se] a number of cases are missing from the analysis of $\dim w^\perp$.) Suppose that $\alpha_{11} = (\nu_{ij}) \in \mathrm{GL}_3(\mathbf{Q}_p)$. Since the dimensions of u_i^\perp should be preserved under automorphisms of \mathcal{L} we have that $\nu_{12} = \nu_{32} = 0$ and hence $\nu_{22} \neq 0$. We now consider certain commutator identities which yield the following relations on ν_{ij} :

- (1) $[u_1, [u_1, u_2]] = 0$ implies $\nu_{11}\nu_{22}\nu_{13} = 0$
- (2) $[[u_2, u_3], u_3] = 0$ implies $\nu_{31}\nu_{22}\nu_{33} = 0$
- (3) $[[u_1, u_2], u_2] = 0$ implies $\nu_{11}\nu_{22}\nu_{23} + \nu_{13}\nu_{22}\nu_{21} = 0$
- (4) $[u_2, [u_2, u_3]] = 0$ implies $\nu_{31}\nu_{22}\nu_{23} + \nu_{33}\nu_{22}\nu_{21} = 0$.

These relations on ν_{ij} together with the fact that $\nu_{11}\nu_{33} - \nu_{13}\nu_{31} \neq 0$ imply that either

$$\alpha(u_i) = \mu_i u_i \bmod \gamma_2(\mathcal{L}) \text{ for } i = 1, 2, 3$$

or

$$\alpha(u_i) = \mu_i u_{4-i} \bmod \gamma_2(\mathcal{L}) \text{ for } i = 1, 2, 3.$$

It is an easy exercise to check that in fact

$$\alpha(u_i) = \mu_i u_i \text{ for } i = 1, 2, 3$$

and

$$\alpha(u_i) = \mu_i u_{4-i} \text{ for } i = 1, 2, 3$$

do determine automorphisms of \mathcal{L} and that α_{22} and α_{33} have the description given in the statement of our proposition. (Consider for example conjugation (in $M_4(\mathbf{Q}_p)$) by matrices in $D_4(\mathbf{Q}_p)$ together with the map $w \rightarrow -^t(J_4 w J_4)$.) Hence we can assume now that α acts trivially on $\mathcal{L}/\gamma_2(\mathcal{L})$. Suppose that

$$\alpha(u_i) = u_i + \lambda_{i1}e_{13} + \lambda_{i2}e_{24} \bmod \gamma_3(\mathcal{L})$$

then since $[u_1, u_3] = 0$ we have that

$$[\alpha u_1, \alpha u_3] = (\lambda_{11} + \lambda_{32})e_{14} = 0$$

i.e. $\lambda_{11} = -\lambda_{32}$. This is the only relation forced on N_1 and it is then a straightforward exercise to check that all such maps define automorphisms of \mathcal{L} . \square

We can generalize this approach to realize a description of

$$\text{Aut}_{\mathbf{Q}_p} U_n^0(\mathbf{Q}_p) / \gamma_{c+1} U_n^0(\mathbf{Q}_p).$$

The key is to consider the sets

$$w^{\perp(l)} = \{\bar{x} \in \mathcal{L}/\gamma_{l+1}(\mathcal{L}) \mid [w, x] \in \gamma_{l+2}(\mathcal{L})\}$$

for $l = 1, \dots, c-1$.

The reductive part H_p of the connected component of G_p in our present example is then just the torus $D_3(\mathbf{Q}_p)$. We consider now the maps $\theta_i : H_p \rightarrow \mathbf{R}$ for $i = 1, 2$ defined in Theorem 2.2.

LEMMA 3.16. *Let $h = (\alpha_{ij}) \in H_p$. Then*

$$\begin{aligned}\theta_1(h) &= |\mu_1|^{-2} |\mu_2|^{-5} |\mu_3|^{-2} \cdot (\min\{|\mu_1|^{-1}, |\mu_3|^{-1}\}) \\ \theta_2(h) &= |\mu_1 \mu_2 \mu_3|^{-3}.\end{aligned}$$

Proof. Recall the definition of $\theta_i(h)$:

$$\theta_i(h) = \mu_{N_i/N_{i+1}} (\{n_i \in N_i/N_{i+1} \mid n_i \tau_i(h) \in M_{3,3-i}(\mathbf{Z}_p)\})$$

where $n_i \tau_i(h) = \alpha'_{1i+1} \cdot \alpha_{i+1i+1}$ and $n_i = (\alpha'_{ij})$. (Note that since the map ψ in Proposition 3.15 is defined over \mathbf{Z} if $\alpha'_{13} \cdot \alpha_{33} \in M_{3,1}(\mathbf{Z}_p)$ then $\alpha'_{23} \cdot \alpha_{33} \in M_{2,1}(\mathbf{Z}_p)$.)

If $i = 1$ then

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \\ \lambda_{31} & -\lambda_{11} \end{pmatrix} \begin{pmatrix} \mu_1 \mu_2 & 0 \\ 0 & \mu_2 \mu_3 \end{pmatrix} \in M_{3,2}(\mathbf{Z}_p)$$

if and only if

$$\begin{aligned}v(\lambda_{i1}) &\geq -v(\mu_1 \mu_2) \text{ for } i = 2, 3 \\ v(\lambda_{i2}) &\geq -v(\mu_2 \mu_3) \text{ for } i = 1, 2 \\ v(\lambda_{11}) &\geq \max\{-v(\mu_1 \mu_2), -v(\mu_2 \mu_3)\}.\end{aligned}$$

Hence

$$\theta_1(h) = |\mu_1 \mu_2|^{-2} |\mu_2 \mu_3|^{-2} \cdot (\min\{|\mu_1 \mu_2|^{-1}, |\mu_2 \mu_3|^{-1}\}).$$

The calculation for $\theta_2(h)$ is straightforward. \square

Hence $\theta_1 : H_p \rightarrow \mathbf{R}$ is not a character of H_p . Nonetheless we can still calculate $\zeta_{L,p}^\wedge(s)$ where L is the lattice spanned by the basis $\{e_{ij} \mid 1 \leq i < j \leq 4\}$. The Assumptions 2.1–2.3 are satisfied so we can apply Theorem 2.2:

$$\zeta_{L,p}^\wedge(s) = \int_{H_p^*} |\det \rho_1(h)|^s |\det \rho_2(h)|^s |\det \rho_3(h)|^s \theta_1(h) \theta_2(h) \mu_H(h)$$

where $\rho_i(h) = \alpha_{ii}$. Then

$$\begin{aligned}\zeta_{L,p}^\wedge(s) &= \int_{(\mu_1, \mu_2, \mu_3) \in (\mathbf{Z}_p^*)^3} |\mu_1|^{3s-5} |\mu_2|^{4s-8} |\mu_3|^{3s-5} \\ &\quad \cdot (\min\{|\mu_1|^{-1}, |\mu_3|^{-1}\}) d\mu_1 d\mu_2 d\mu_3 \\ &= \frac{(1 + p^{-3s+5})}{(1 - p^{-4s+8})(1 - p^{-3s+5})(1 - p^{-6s+11})}\end{aligned}$$

where $d\mu_i$ is the Haar measure on the multiplicative group \mathbf{Q}_p^* normalized such that $\mu_i(\mathbf{Z}_p^*) = 1$. (Note that $\mu_i(p^n \mathbf{Z}_p \setminus p^{n+1} \mathbf{Z}_p) = \mu_i(p^n \mathbf{Z}_p^*) = 1$.)

Hence $\zeta_{L,p}^\wedge(s)$ satisfies the functional equation

$$\zeta_{L,p}^\wedge(s)|_{p \rightarrow p^{-1}} = -p^{-10s+19} \zeta_{L,p}^\wedge(s).$$

Calculations of some higher dimensional examples hint at the fact that we still have a functional equation despite the fact that the θ_i are not in general characters.

4. Zeta functions for algebraic groups over a global number field. In this section we consider a linear algebraic group G defined over a field K , where K is a finite extension of \mathbf{Q} , together with a K -rational representation

$$\rho : G \rightarrow \mathrm{GL}_n.$$

For each prime \mathfrak{p} of K recall from Definition 0.3 (ii) that

$$Z_{G,\rho,\mathfrak{p}}(s) = \int_{G_{\mathfrak{p}}^+} |\det \rho(g)|_{\mathfrak{p}}^s \mu_G(g)$$

where $G_{\mathfrak{p}}^+ = \rho^{-1}(\rho(G(K_{\mathfrak{p}})) \cap \mathrm{M}_n(\mathcal{O}_{K_{\mathfrak{p}}}))$. We will prove that for almost all primes \mathfrak{p} of K , $G(K_{\mathfrak{p}})$ and the representation $\rho : G(K_{\mathfrak{p}}) \rightarrow \mathrm{GL}_n(K_{\mathfrak{p}})$ satisfy the Assumptions 2.1, 2.2 and 2.3 of section 2. This implies then that for almost all primes \mathfrak{p} we can replace $Z_{G,\rho,\mathfrak{p}}(s)$ by an integral over the connected component of the reductive part of G as detailed in Theorem 2.2.

We shall keep track in a note at the end of each lemma which primes we are excluding. We begin with proving that Assumption .1 holds for almost all primes \mathfrak{p} .

LEMMA 4.1. *For almost all primes \mathfrak{p} , $G(\mathcal{O}_{K_{\mathfrak{p}}})$ maps onto $G(K_{\mathfrak{p}})/G_0(K_{\mathfrak{p}})$ where G_0 is the connected component of G .*

Proof. There exists a finite K -algebraic group F such that $G(K') = G_0(K')F(K')$ for any field extension K' of K (see [BS] Lemma 5.11). There exists a finite extension L of K such that $F(L') = F(L)$ for any $L' \geq L$. Let $(a_{ij}) \in F(L)$ then, for almost all primes \mathfrak{p} of K , either $a_{ij} \notin K_{\mathfrak{p}}$ or $a_{ij} \in \mathcal{O}_{K_{\mathfrak{p}}}$. Since $F(L)$ is finite, for almost all primes \mathfrak{p} ,

$$F(K_{\mathfrak{p}}) = F(L) \cap \mathrm{GL}_n(K_{\mathfrak{p}}) \leq \mathrm{GL}_n(\mathcal{O}_{K_{\mathfrak{p}}}).$$

This proves the lemma. □

Note. We exclude at this stage primes \mathfrak{p} for which an element of the finite group $F(K_{\mathfrak{p}})$ is not integral.

The following fact that choosing an equivalent K -rational representation for G does not affect $Z_{G,\rho,\mathfrak{p}}(s)$ for almost all primes \mathfrak{p} will be important here and in §6.

LEMMA 4.2. *Let $\rho' : G \rightarrow \mathrm{GL}_n$ be an equivalent K -rational representation of ρ ; i.e., there exists $A \in \mathrm{GL}_n(K)$ such that $\rho'(x) = A\rho(x)A^{-1}$ for all $x \in G(K)$. Then for almost all primes \mathfrak{p} of K*

$$Z_{G,\rho',\mathfrak{p}}(s) = Z_{G,\rho,\mathfrak{p}}(s).$$

Proof. The integrand $|\det \rho(g)|^s$ is independent of a choice of equivalent representation. All we have to worry about is the subset $G_{\mathfrak{p}}^+$ over which we are integrating. For almost all primes \mathfrak{p} , $A \in \mathrm{GL}_n(\vartheta_{K_{\mathfrak{p}}})$. In this situation if $\rho(g) \in \mathrm{M}_n(\vartheta_{K_{\mathfrak{p}}})$ then

$$\rho'(g) = A\rho(g)A^{-1} \in A\mathrm{M}_n(\vartheta_{K_{\mathfrak{p}}})A^{-1} = \mathrm{M}_n(\vartheta_{K_{\mathfrak{p}}})$$

and conversely. Hence $Z_{G,\rho',\mathfrak{p}}(s) = Z_{G,\rho,\mathfrak{p}}(s)$. \square

Note. Here we are excluding primes \mathfrak{p} for which A or $A^{-1} \notin \mathrm{M}_n(\vartheta_{K_{\mathfrak{p}}})$.

We show next that we can choose an equivalent representation satisfying Assumption 2.2. Recall that $N(K)$ is the unipotent radical of $G(K)$ and $H(K)$ is the connected component of the reductive part of $G(K)$.

LEMMA 4.3. *There exists an equivalent K -rational representation ρ' of ρ such that ρ' decomposes into block form where $\rho'|_{H(K)}$ is block diagonal and $\rho'|_{N(K)}$ is unitriangular.*

Proof. We are required to decompose $V = K^n$ into a direct sum $V = U_1 \oplus \cdots \oplus U_c$ of $H(K)$ -stable subspaces U_i such that $N(K)$ acts trivially on V_i/V_{i+1} where $V_i = U_i \oplus \cdots \oplus U_c$. Let $0 \neq v \in V$ be a fixed point of the action of $N(K)$ on V , the existence of which is guaranteed by Lie-Kolchin. Set $U_c = v^{G(K)}$. Then U_c is $H(K)$ -stable and $N(K)$ acts trivially on U_c . There exists an H -stable splitting $V = W \oplus U_c$. Now proceed by induction. \square

Assumption 2.3 will follow as a corollary of the following result:

LEMMA 4.4. *Let K be a field of characteristic zero and G_1 and $G_2 \leq \mathrm{GL}_n(K)$ be K -linear algebraic groups. Suppose*

$$\varphi : G_1 \rightarrow G_2$$

is a K -rational epimorphism with $\ker \varphi$ unipotent. Then there exists a polynomial section defined over K

$$\psi : G_2 \rightarrow G_1$$

such that $\varphi \circ \psi = \text{id}$.

Proof. We assume first that G_1 and G_2 are both unipotent. Denote by $\mathcal{L}(G_i)$ the Lie algebra of G_i . The homomorphism $\mathcal{L}(\varphi) = \log|_{G_2} \circ \varphi \circ \exp|_{\mathcal{L}(G_1)} : \mathcal{L}(G_1) \rightarrow \mathcal{L}(G_2)$ splits by a linear transformation $\varphi_1 : \mathcal{L}(G_2) \rightarrow \mathcal{L}(G_1)$. Now $\psi = \exp|_{\mathcal{L}(G_1)} \circ \varphi_1 \circ \log|_{G_2}$ defines a section with the property that $\varphi \circ \psi = \text{id}$. Since G_i is unipotent, $\exp|_{\mathcal{L}(G_1)}$ and $\log|_{G_2}$ are both polynomial maps defined over K and hence ψ is a polynomial section defined over K .

In the general case we write $G_i = N_i \rtimes H_i$ as a semi-direct product of its unipotent radical N_i and its reductive part H_i . We can choose H_2 such that $\varphi|_{H_1}$ induces an isomorphism from H_1 to H_2 . Let $\psi_1|_{H_2} = \varphi^{-1}|_{H_2}$ which, as a morphism of algebraic groups, is defined by polynomials over K . Let $\psi_2 : N_2 \rightarrow N_1$ be the polynomial section guaranteed by the first part of this proof. We define $\psi : G_2 \rightarrow G_1$ by $\psi(n_2 h_2) = \psi_2(n_2) \psi_1(h_2)$. Then ψ is a polynomial section defined over K with $\varphi \circ \psi = \text{id}$. \square

COROLLARY 4.5. *For almost all primes \mathfrak{p} of K , if $\bar{g} \in (G(K_{\mathfrak{p}})/N_i(K_{\mathfrak{p}}))^+$ (where the integral points are taken with respect to the representation φ_i of $G(K_{\mathfrak{p}})/N_i(K_{\mathfrak{p}})$ defined in §2) then there exists $g \in G(K_{\mathfrak{p}})^+$ such that $gN_i(K_{\mathfrak{p}}) = \bar{g}$.*

Proof. Let $\varphi : \rho(G(K)) \rightarrow \varphi_i(G(K)/N_i(K))$ be the natural map with kernel the unipotent group $\rho(N_i(K))$. By the previous lemma there exists a polynomial section defined over K , $\psi : \varphi_i(G(K)/N_i(K)) \rightarrow \rho(G(K))$ such that if $\bar{g} \in \varphi_i(G(K)/N_i(K))$ and $g = \psi(\bar{g})$ then $\bar{g} = \varphi(g)$. This polynomial section extends for each prime \mathfrak{p} to a section $\psi : \varphi_i(G(K_{\mathfrak{p}})/N_i(K_{\mathfrak{p}})) \rightarrow \rho(G(K_{\mathfrak{p}}))$. For almost all primes \mathfrak{p} this polynomial section is defined over $\vartheta_{K_{\mathfrak{p}}}$ and hence if $\bar{g} \in \varphi_i(G(K_{\mathfrak{p}})/N_i(K_{\mathfrak{p}})) \cap M_n(\vartheta_K)$ then $g = \psi(\bar{g}) \in \rho(G(K_{\mathfrak{p}})) \cap M_n(\vartheta_K)$. This proves the corollary. \square

Note. The primes excluded at this stage are those for which the coefficients of the polynomials defining the section ψ are not integers in the localization $K_{\mathfrak{p}}$.

Lemmas 4.1, 4.3 and Corollary 4.5 together with the results of section 2 then give us the following:

COROLLARY 4.6. *Let G be a K -algebraic group and*

$$\rho : G \rightarrow \text{GL}_n$$

a K -rational representation where K is a finite extension of \mathbf{Q} . Then for almost all primes \mathfrak{p} ,

$$(4.1) \quad Z_{G,\rho,\mathfrak{p}}(s) = \int_{H_{\mathfrak{p}}^+} |\det h|^s \prod_{i=1}^{c-1} \theta_i(h) \mu_H(h)$$

where $H_{\mathfrak{p}}^+ = \rho^{-1}(\rho(H(K_{\mathfrak{p}})) \cap \mathbf{M}_n(\vartheta_{K_{\mathfrak{p}}}))$ and θ_i are the functions on $H(K_{\mathfrak{p}})$ defined in section 2.

In the next section we turn to the problem of evaluating an integral (4.1) for a connected reductive algebraic group H . We shall have to impose further conditions on the reductive group H to make this calculation.

5. An explicit finite form and a functional equation. Let G be a connected reductive linear algebraic group defined over k where k is a finite extension of \mathbf{Q}_p . Denote by q the order of the residue field of k . Let $\rho : G \rightarrow \mathrm{GL}_n$ be a faithful k -rational representation. In §2 we were left having to consider integrals of the form $Z_{G,\rho,\beta,\theta}(s)$ as defined in Definition 0.4 (i). At present we can only deal with the case that $\theta : H \rightarrow \mathbf{R}$ is a character on H . Note that, by Theorem 2.3, this is the case when G is the automorphism group associated with a class two nilpotent group or a nilpotent group free in some variety. We therefore restrict our analysis to the following integrals:

$$Z_{G,\rho,\beta_1,\beta_2}(s) = \int_{G^+} |\beta_1(g)|^s |\beta_2(g)| \mu_G(g)$$

where $\beta_1, \beta_2 \in \mathrm{Hom}(G, \mathbf{G}_m)$ are two characters on G . As in section 2 we shall introduce various assumptions on our reductive group such that we can apply the methodology of Igusa to calculate $Z_{G,\rho,\beta_1,\beta_2}(s)$. Our setting requires a slight generalization of Igusa's calculation. Unlike section 4 it will not be the case that all these assumptions are true for almost all primes when we start with an algebraic group defined over a global number field. We shall make some comments in this direction and hope in a future paper to remove the assumptions we need at the moment to make Igusa's calculation. But we content ourselves in this paper with defining a class of algebraic groups for which an explicit finite form for $Z_{G,\rho,\beta_1,\beta_2}(s)$ exists.

We shall be reasonably sparing with details since these can be found in Igusa's paper [I].

We can write $G = S.G'$ where S is a central k -torus and G' , the derived group, is a connected semisimple algebraic group and $S \cap G' \cong \mu_m$ where μ_m denotes the group of m -th roots of unity.

Assumption 5.1. $S = \mathbf{G}_m$ i.e., the maximal central torus of G is one-dimensional.

This is the first nontrivial case since if S is finite then $Z_{G,\rho,\beta_1,\beta_2}(s)$ is constant. Note that this assumption followed from Igusa's assumption that G is an irreducible subgroup of GL_n (not contained in SL_n). However we shall want to consider non-irreducible representations $\rho : G \rightarrow \mathrm{GL}_n$.

Under this assumption, $\mathrm{Hom}(G, \mathbf{G}_m)$ is generated by a single element f satisfying $f(\tau) = \tau^{\epsilon m}$ for every $\tau \in S$ and $\ker f = G'$ where $\epsilon \in \{-1, 1\}$. Later we shall choose a generator, i.e. choose ϵ , to make our calculation smoother. We can write the characters $\beta_i = f^{r_i}$ for some $r_i \in \mathbf{Z}$ and hence we have

LEMMA 5.1.

$$Z_{G,\rho,\beta_1,\beta_2}(s) = Z_{G,\rho,f}(r_1 s + r_2)$$

where $Z_{G,\rho,f}(s) = \int_{G^+} |f(g)|^s \mu_G(g)$.

So under Assumption 5.1 we can focus our attention on $Z_{G,\rho,f}(s)$.

The key to calculating $Z_{G,\rho,f}(s)$ is the p -adic Bruhat decomposition and the expression for the measure of double cosets in this decomposition as the distance between chambers in the associated building. To apply this decomposition we need at present to make the following assumption:

Assumption 5.2. The maximal torus T splits over k .

In this situation we call G k -split. (Note that, following the work of Bruhat and Tits, the notion of a p -adic Bruhat decomposition exists for a non-split group G which should allow us, we hope, to carry out the following calculation in this setting eventually.) T is unique up to conjugation in $G(k)$ and hence contains S . Since T splits over k it is k -isomorphic to $(\mathbf{G}_m)^{\dim(T)}$. Let $\phi : T \rightarrow (\mathbf{G}_m)^{\dim T}$ denote such an isomorphism.

When G is k -split, the nontrivial minimal closed unipotent subgroups of G normalized by T are isomorphic to \mathbf{G}_a . The conjugation action of T is mapped by this isomorphism to an action of T on \mathbf{G}_a of the form

$$x \mapsto \alpha(t)x$$

where $\alpha \in \mathrm{Hom}(T, \mathbf{G}_m)$. The elements $\alpha \in \mathrm{Hom}(T, \mathbf{G}_m)$ thus obtained are all distinct nonzero and finite in number. They form a reduced root system Φ in the subspace of $V = \mathrm{Hom}(T, \mathbf{G}_m) \otimes_{\mathbf{Z}} \mathbf{R}$ that they generate. The elements of Φ are called the *roots of G relative to T* . For each $\alpha \in \Phi$, let U_α be the corresponding minimal unipotent subgroup and $\theta_\alpha : \mathbf{G}_a \rightarrow U_\alpha$ denote a k -isomorphism such that

$$t\theta_\alpha(u)t^{-1} = \theta_\alpha(\alpha(t)u).$$

Let π denote a fixed uniformizing parameter for ϑ_k . We then make the following:

Assumption 5.3. The groups G and T together with the isomorphisms $\phi : T \rightarrow (\mathbf{G}_m)^{\dim(T)}$ and $\theta_\alpha : \mathbf{G}_a \rightarrow U_\alpha$ for each $\alpha \in \Phi$ have good reduction mod π . In this setting we shall say (after Igusa) that G has *very good reduction mod π* .

We refer to [B] and [PR] § 3.3 for an explanation of what it means for a group and a homomorphism of groups to have good reduction. This assumption, which is perhaps the most technical of our assumptions, is satisfied for almost all primes \mathfrak{p} if we consider an algebraic group G defined over a global number field K and take its $K_{\mathfrak{p}}$ -points $G(K_{\mathfrak{p}})$ (see Proposition 3.20 [PR]).

The finite Weyl group W of G relative to T is defined as $W = N(k)/T(k)$ where N denotes the normalizer of T in G . The Weyl group W is isomorphic to the Weyl group of the root system Φ . One consequence of Assumption 5.3 is that we can choose coset representatives g_w for every w in W from $N(\vartheta_k)$. An explanation of this can be found in [I] II.2.

Let Ξ denote $\text{Hom}(\mathbf{G}_m, T)$ and $V^* = \Xi \otimes_{\mathbf{Z}} \mathbf{R}$. Then Ξ is dual to $\text{Hom}(T, \mathbf{G}_m)$ under the natural pairing

$$\text{Hom}(T, \mathbf{G}_m) \times \text{Hom}(\mathbf{G}_m, T) \rightarrow \mathbf{Z}$$

$$(\alpha, \xi) \mapsto \langle \alpha, \xi \rangle$$

where $\alpha(\xi(\tau)) = \tau^{\langle \alpha, \xi \rangle}$ for every τ in \mathbf{G}_m . The Weyl group W can then be embedded in $\text{GL}(V^*)$ since it acts on the coroots Φ^* of Φ which can be identified with elements of V^* under this pairing. We now extend W to a group \mathcal{W} of affine linear transformations of V^* as follows: For each $\xi \in \Xi$ define $t_\xi(x) = x + \xi$ for each $x \in V^*$ and set

$$\mathcal{W} = W \cdot \{t_\xi \mid \xi \in \Xi\},$$

the semi-direct product of W and translations by Ξ . The law of multiplication in \mathcal{W} is defined by

$$(w_1 t_{\xi_1})(w_2 t_{\xi_2}) = w_1 w_2 t_\xi \text{ where } \xi = w_2^{-1}(\xi_1) + \xi_2.$$

\mathcal{W} is called the affine Weyl group of G relative to T . It is isomorphic to the group $N(k)/T(\vartheta_k)$ via the map $w t_\xi \mapsto g_w \xi(\pi) T(\vartheta_k)$. (Note that the coroots Φ^* span V^* if and only if G is semisimple; hence \mathcal{W} is in general larger than the affine Weyl group of the root system Φ . Under Assumption 5.1 the quotient of \mathcal{W} by the affine Weyl group of Φ is isomorphic to \mathbf{Z} .)

Choose a basis $\Phi_0 = \{\alpha_1, \dots, \alpha_l\}$ for Φ so that Φ^+ and Φ^- , the set of positive and negative roots, are defined. Recall that the finite Weyl group W acts simply

transitively on the set of all bases for Φ . The functional equation that follows from Igusa's explicit form for $Z_{G,\rho,f}(s)$ depends on the freedom here to choose a different basis for Φ . This choice of basis fixes a fundamental cell \mathcal{C} in V^* defined by

$$\mathcal{C} = \{x \in V^* \mid 0 < \langle \alpha, x \rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

Define $U^+ = \prod_{\alpha \in \Phi^+} \theta_\alpha(\mathbf{G}_a)$ and U^- similarly. Then, setting

$$\mathcal{B} = U^+(\pi\vartheta_k)T(\vartheta_k)U^-(\vartheta_k),$$

\mathcal{B} is an Iwahori subgroup and we have the following p -adic Bruhat decomposition of $G(k)$:

PROPOSITION 5.2. (i) $G(k)$ can be written as a disjoint union of double cosets of \mathcal{B} as follows:

$$\begin{aligned} G(k) &= \bigcup_{wt_\xi \in \mathcal{W}} \mathcal{B}g_w\xi(\pi)\mathcal{B}; \\ \text{and } G(\vartheta_k) &= \bigcup_{w \in W} \mathcal{B}g_w\mathcal{B}. \end{aligned}$$

(ii) Define the function $\lambda(wt_\xi)$ by

$$\begin{aligned} q^{\lambda(wt_\xi)} &= \text{card}(\mathcal{B}g_w\xi(\pi)\mathcal{B}/\mathcal{B}) \\ &= \mu_G(\mathcal{B}g_w\xi(\pi)\mathcal{B})/\mu_G(\mathcal{B}), \end{aligned}$$

the index of \mathcal{B} in the double coset $\mathcal{B}g_w\xi(\pi)\mathcal{B}$. Then for each $\xi \in \Xi$ there is a unique element $w_\xi \in W$ such that the function $w \mapsto \lambda(wt_\xi)$ on W attains its minimum at w_ξ with value

$$(5.1) \quad \lambda(w_\xi t_\xi) = \sum_{\alpha \in w_\xi^{-1}(\Phi^+)} \langle \alpha, \xi \rangle - \lambda(w_\xi).$$

The element w_ξ has the property that for all $w \in W$

$$(5.2) \quad \lambda(ww_\xi t_\xi) = \lambda(w) + \lambda(w_\xi t_\xi).$$

The formulas (5.1) and (5.2) for $\lambda(wt_\xi)$, established by Iwahori and Matsumoto [IwM], can be understood from the interpretation of $\lambda(wt_\xi)$ as the number of hyperplanes in V^* separating the fundamental cell \mathcal{C} from its image $\sigma(\mathcal{C})$ under the action of $\sigma = wt_\xi \in \mathcal{W}$. For more details we refer the reader to [IwM] and

[I]. The expression for $\lambda(wt_\xi)$ and the decomposition of $G(k)$ are crucial to the calculation of $Z_{G,\rho,f}(s)$.

Since $\mathcal{B} \leq G(\vartheta_k)$ and $g_w \in N(\vartheta_k)$, then $\mathcal{B}g_w\xi(\pi)\mathcal{B} \subseteq G^+$ if and only if $\rho(\xi(\pi)) \in M_n(\vartheta_k)$. Set

$$\Xi^+ = \{\xi \in \Xi \mid \rho(\xi(\pi)) \in M_n(\vartheta_k)\}$$

and for each $w \in W$

$$\Xi_w = \{\xi \in \Xi \mid w\xi = w\}.$$

Then we can make the first inroads on the calculation of $Z_{G,\rho,f}(s)$. Note that since $\mathcal{B}g_w\mathcal{B} \leq G(\vartheta_k)$, the integrand $|f(g)|^s$ is constant on $\mathcal{B}g_w\xi(\pi)\mathcal{B}$ and takes the value $|f(\xi(\pi))|^s$. So

$$\begin{aligned} Z_{G,\rho,f}(s) &= \mu(\mathcal{B}) \cdot \sum_{w \in W} \sum_{\xi \in \Xi^+} q^{\lambda(wt_\xi)} |f(\xi(\pi))|^s \\ &= \mu(\mathcal{B}) \cdot \sum_{w' \in W} \sum_{w \in W} \sum_{\xi \in \Xi_w^+} q^{\lambda(w'wt_\xi)} |f(\xi(\pi))|^s \\ &= (\mu(\mathcal{B}) \cdot \sum_{w' \in W} q^{\lambda(w')}) \left(\sum_{w \in W} \sum_{\xi \in \Xi_w^+} q^{\lambda(wt_\xi)} |f(\xi(\pi))|^s \right) \end{aligned}$$

where $\Xi_w^+ = \Xi_w \cap \Xi^+$. But $\mu(\mathcal{B}) \cdot \sum_{w \in W} q^{\lambda(w)} = \mu_G(G(\vartheta_k)) = 1$ by our normalization of the Haar measure μ_G . Hence

$$Z_{G,\rho,f}(s) = \sum_{w \in W} \sum_{\xi \in \Xi_w^+} q^{\lambda(wt_\xi)} |f(\xi(\pi))|^s.$$

By Proposition 5.2 (ii), for $\xi \in \Xi_w$

$$\begin{aligned} \lambda(wt_\xi) &= \sum_{\alpha \in w^{-1}(\Phi^+)} \langle \alpha, \xi \rangle - \lambda(w) \\ &= \sum_{\alpha \in \Phi^+} \langle \alpha, w\xi \rangle - \lambda(w) \end{aligned}$$

since $\langle \cdot, \cdot \rangle$ is invariant under the action of W . We set

$$(5.3) \quad \prod_{\alpha \in \Phi^+} \alpha = \prod_{i=1}^l \alpha_i^{a_i}$$

where a_1, \dots, a_l are positive integers depending only on Φ . Then, since $|f(w(\xi)(\pi))| =$

$|f(g_w \xi(\pi) g_w^{-1})| = |f(\xi(\pi))|$, we have

$$(5.4) \quad \begin{aligned} Z_{G,\rho,f}(s) &= \sum_{w \in W} \sum_{\xi \in w\Xi_w^+} q^{\sum_{\alpha \in \Phi^+} (\langle \alpha, w\xi \rangle - \lambda(w))} |f(w(\xi)(\pi))|^s \\ &= \sum_{w \in W} q^{-\lambda(w)} \sum_{\xi \in w\Xi_w^+} q^{a_1 \langle \alpha_1, \xi \rangle + \dots + a_l \langle \alpha_l, \xi \rangle} |f(\xi(\pi))|^s. \end{aligned}$$

We turn now to an analysis of the representation ρ in order to understand the set $w\Xi_w^+$. Since G is a reductive group, the representation ρ is completely reducible i.e., it is equivalent over $\mathrm{GL}_n(k)$ to a direct sum of irreducible representations. Our Assumption 5.2 implies that we can choose an equivalent representation of ρ such that the maximal k -split torus T consists of all diagonal matrices in G . We shall in fact suppose that ρ is such an equivalent representation:

Assumption 5.4. There exist k -rational irreducible representations $\rho_i : G \rightarrow \mathrm{GL}_{n_i}$ ($i = 1, \dots, r$) such that $\rho(g)$ is the diagonal block matrix with $\rho_1(g), \dots, \rho_r(g)$ on the diagonal for each $g \in G(k)$

$$\rho(g) = \begin{pmatrix} \rho_1(g) & & 0 \\ & \ddots & \\ 0 & & \rho_r(g) \end{pmatrix},$$

and the maximal k -split torus $\rho(T(k))$ consists of all diagonal matrices in $\rho(G(k))$.

Note that if we begin with a representation ρ over K a finite extension of \mathbf{Q} then there exists an equivalent representation ρ' over $\mathrm{GL}_n(K)$ such that ρ' satisfies Assumption 5.4. But by Lemma 4.2, for almost all primes \mathfrak{p} of K , $Z_{G,\rho',\mathfrak{p}}(s) = Z_{G,\rho,\mathfrak{p}}(s)$. Hence Assumption 5.4 can be dropped in the global setting by excluding finitely many primes.

We are interested in knowing when $\rho(t) = \rho(\xi(\pi)) \in \rho(T(k)) \cap \mathrm{M}_n(\vartheta_k)$. Under our Assumption 5.4, $\rho(T(k))$ is diagonal. The diagonal entries of $t \in T(k)$ are given by the weights $\omega_{ij} \in \mathrm{Hom}(T, \mathbf{G}_m)$ ($j = 1, \dots, n_i$) of the representations ρ_i . If we denote by $\omega_i \in \mathrm{Hom}(T, \mathbf{G}_m)$ the dominant weight of the irreducible representation $g \mapsto {}^t \rho_i(g)^{-1}$, the contragredient representation of ρ_i , then there exist l -tuples $\mathbf{c}(j, i) = (c_1(j, i), \dots, c_l(j, i)) \in \mathbf{N}^l$ ($j = 1, \dots, n_i$) such that

$$\omega_{ij}(t) = \omega_i^{-1}(t) \cdot \prod_{k=1}^l \alpha_k^{c_k(j, i)}(t).$$

For each i there exists a unique j such that $\mathbf{c}(j, i) = (0, \dots, 0)$. (For a reference see [S] VII and VIII.) For each $i = 1, \dots, r$ there exists $m_i \in \mathbf{Z}$ such that $\omega_i(\tau)^m = f(\tau)^{-m_i}$ for every $\tau \in S$. We shall make a final assumption (Assumption 5.5) that will in fact imply that $m_i > 0$ for all $i = 1, \dots, r$. Since the representation

ρ is faithful, the greatest common divisor of m_1, \dots, m_r , which we denote by (m_1, \dots, m_r) , is equal to 1. Choose $\lambda_1, \dots, \lambda_r \in \mathbf{Z}$ such that $\lambda_1 m_1 + \dots + \lambda_r m_r = 1$.

Recall that $\{\alpha_1, \dots, \alpha_l\}$ was our choice of basis for the root system Φ . We set $\alpha_0 = f|_T$ where f is the generator we have chosen for $\text{Hom}(G, \mathbf{G}_m)$. Then $\alpha_0, \alpha_1, \dots, \alpha_l$ form a set of free generators for $\text{Hom}(Q, \mathbf{G}_m)$ where $Q = T/S \cap G'$ and $S \cap G' \cong \mu_m$, the group of m th roots of unity. This follows from the fact that

$$\bigcap_{i=0}^l \ker \alpha_i = \ker \alpha_0 \cap \bigcap_{i=1}^l \ker \alpha_i = G' \cap S.$$

Let $\xi_0, \xi_1, \dots, \xi_l$ denote the elements of $\text{Hom}(\mathbf{G}_m, Q)$ satisfying $\langle \alpha_i, \xi_j \rangle = \delta_{ij}$ for $0 \leq i, j \leq l$ then $\{\xi_0, \xi_1, \dots, \xi_l\}$ is a set of free generators of $\text{Hom}(\mathbf{G}_m, Q)$.

For each $i = 1, \dots, r$, $\omega_i^m \in \text{Hom}(Q, \mathbf{G}_m)$ and hence can be written uniquely as a \mathbf{Z} -linear combination of the basis $\alpha_0, \dots, \alpha_l$. For $\tau \in S$, $\omega_i^m(\tau) = \alpha_0^{-m_i}(\tau)$ and $\alpha_j(\tau) = 1$ for $1 \leq j \leq l$. Hence

$$(5.5) \quad \omega_i^m = \alpha_0^{-m_i} \cdot \prod_{j=1}^l \alpha_j^{b_j(i)}$$

for some $b_1(i), \dots, b_l(i) \in \mathbf{Z}$. In fact $b_j(i) \geq 0$ for $1 \leq j \leq l$ and $1 \leq i \leq r$ since ω_i is a dominant weight. As Igusa points out, this can be deduced from the fact that all entries of the Cartan matrix of a complex simple Lie algebra are positive. Note however that in our setting $b_j(i)$ can be zero. This corresponds to those nonfaithful representations ρ_i . For such a representation $\{\alpha_j | b_j(i) = 0\}$ is a basis for a root system of the kernel of ρ_i .

$$\text{LEMMA 5.3. } \Xi = \text{Hom}(\mathbf{G}_m, T) = \xi_0^{m\mathbf{Z}} \cdot \prod_{j=1}^l (\xi_0^{\lambda_1 b_j(1) + \dots + \lambda_r b_j(r)} \xi_j)^{\mathbf{Z}}.$$

Proof. By duality Ξ is a subgroup of $\text{Hom}(\mathbf{G}_m, Q)$ with $\mathbf{Z}/m\mathbf{Z}$ as the factor group. Each element ξ of $\text{Hom}(\mathbf{G}_m, Q)$ can be expressed uniquely as $\xi = \prod_{j=0}^l \xi_j^{e_j}$ for some $e_0, e_1, \dots, e_l \in \mathbf{Z}$. Then ξ is in Ξ if and only if $\langle \alpha, \xi \rangle \in \mathbf{Z}$ for all $\alpha \in \text{Hom}(T, \mathbf{G}_m)$. Set $\omega = \omega_1^{\lambda_1} \dots \omega_r^{\lambda_r}$, where $\lambda_1, \dots, \lambda_r$ were chosen above. Then $S \cap \ker \omega = \bigcap_{i=1}^l \ker \alpha_i \cap \ker \omega = 1$. Hence $\text{Hom}(T, \mathbf{G}_m)$ is generated by $\alpha_1, \dots, \alpha_l$ and ω . Since $\langle \alpha_i, \xi \rangle \in \mathbf{Z}$ for $i = 1, \dots, r$, ξ is in Ξ if and only if $\langle \omega, \xi \rangle \in \mathbf{Z}$. This is the case if and only if

$$-(\lambda_1 m_1 + \dots + \lambda_r m_r) e_0 + \sum_{j=1}^l (\lambda_1 b_j(1) + \dots + \lambda_r b_j(r)) e_j \equiv 0 \pmod{m}.$$

Since $\lambda_1 m_1 + \dots + \lambda_r m_r = 1$ the lemma follows. \square

We define $c_1, \dots, c_l \in \mathbf{Z}$ by

$$c_j = \lambda_1 b_j(1) + \dots + \lambda_r b_j(r).$$

Let C denote the positive Weyl chamber relative to the choice of basis Φ_0 defined by

$$C = \{x \in V^* \mid 0 < \langle \alpha, x \rangle \text{ for all } \alpha \in \Phi^+\}$$

and \overline{C} its closure in V^* . Igusa established previously (see [I] II.3) that the set $w\Xi_w^+$ has the following description:

$$(5.6) \quad w\Xi_w^+ = \{\xi \in \Xi^+ \cap \overline{C} \mid \langle \alpha_i, \xi \rangle > 0 \text{ if } \alpha_i \in w(\Phi^-)\}$$

The final piece in the jigsaw is to analyze the set $\Xi^+ \cap \overline{C}$. By definition of \overline{C} we have

$$\Xi \cap \overline{C} = \xi_0^{m\mathbf{Z}} \cdot \prod_{j=1}^l (\xi_0^{c_j} \xi_j)^{\mathbf{N}}.$$

LEMMA 5.4. *Let $\xi \in \Xi \cap \overline{C}$. Then $\xi \in \Xi^+ \cap \overline{C}$ if and only if $\langle \omega_i^{-1}, \xi \rangle \geq 0$ for $i = 1, \dots, r$.*

Proof. The element $\xi = \xi_0^{m\mathbf{Z}} \cdot \prod_{j=1}^l (\xi_0^{c_j} \xi_j)^{\mathbf{e}_j}$ of Ξ is contained in Ξ^+ if and only if $\omega_{ik}(\xi(\pi)) \in \vartheta_k$, i.e. $\langle \omega_{ik}, \xi \rangle \geq 0$, for each $i = 1, \dots, r$ and $k = 1, \dots, n_i$. Since, for each $i = 1, \dots, r$, $\omega_i^{-1} = \omega_{ik}$ for some k , one direction of the lemma is clear. Suppose then that $\langle \omega_i^{-1}, \xi \rangle \geq 0$ for $i = 1, \dots, r$. Then, for each $i = 1, \dots, r$ and $k = 1, \dots, n_i$,

$$\langle \omega_{ik}, \xi \rangle = \sum_{j=1}^l c_j(k, i) e_j + \langle \omega_i^{-1}, \xi \rangle \geq 0$$

since $\mathbf{e}(k, i) \in \mathbf{N}^l$ and $e_j \geq 0$ for $1 \leq j \leq l$. This completes the proof of the lemma. \square

This lemma shows why it is the dominant weight of the contragredient representation which dictates whether $\xi \in \Xi^+ \cap \overline{C}$.

To complete the description of $\Xi^+ \cap \overline{C}$ in a similar fashion to Igusa's calculation we need to make one final assumption concerning the irreducible representations ρ_i . To state this assumption we extend from semisimple groups to reductive groups the definition of what it means for a weight ω_1 to dominate a weight ω_2 .

Definition 5.5. Let ω_1 and ω_2 be weights of our reductive algebraic group G . With respect to some choice of generator f for $\text{Hom}(G, \mathbf{G}_m)$ and denoting $\alpha_0 = f|_T$ we can write

$$\omega_i^m = \alpha_0^{-m_i} \cdot \prod_{j=1}^l \alpha_j^{b_j(i)}.$$

We say that ω_1 *dominates* ω_2 if either

- (1) $m_1 m_2 > 0$ and $b_j(1)/m_1 \geq b_j(2)/m_2$ for each $j = 1, \dots, l$; or
- (2) $m_1 = m_2 = 0$ and $b_j(1) \geq b_j(2)$ for each $j = 1, \dots, l$. If the ω_i are dominant weights of representations ρ_i , we say that ρ_1 *dominates* ρ_2 .

Note that if $m_1 = m_2 = 0$ then ω_i is in fact a weight of the semisimple group G' and our definition is the same as the definition for semisimple groups.

We then make the following:

Assumption 5.5. There exists $i_0 \in \{1, \dots, r\}$ such that ${}^t \rho_{i_0}^{-1}$ dominates ${}^t \rho_i^{-1}$ for each $i = 1, \dots, r$. Without loss of generality we may suppose that $i_0 = 1$.

Note that since $m_i \neq 0$ for some i , this assumption implies that $m_i \neq 0$ for all i and without loss of generality we can choose a generator f for $\text{Hom}(G, \mathbf{G}_m)$ such that $m_i > 0$ for all i .

In fact Assumption 5.5 has the following equivalent reformulation:

LEMMA 5.6. ${}^t \rho_1^{-1}$ dominates ${}^t \rho_i^{-1}$ for each $i = 1, \dots, r$ if and only if, for all $\xi \in \Xi \cap \overline{C}$, if $\langle \omega_1^{-1}, \xi \rangle \geq 0$ then $\langle \omega_i^{-1}, \xi \rangle \geq 0$ for all $i = 1, \dots, r$.

Proof. Suppose that ${}^t \rho_1^{-1}$ dominates ${}^t \rho_i^{-1}$ for each $i = 1, \dots, r$. Let

$$\xi = \xi_0^{me_0} \cdot \prod_{j=1}^l (\xi_0^{c_j} \xi_j)^{e_j} \in \Xi \cap \overline{C}$$

and suppose that

$$\langle \omega_1^{-1}, \xi \rangle = 1/m \left(m_1 \left(me_0 + \sum_{j=1}^l c_j e_j \right) - \sum_{j=1}^l b_j(1) e_j \right) \geq 0.$$

Then $(me_0 + \sum_{j=1}^l c_j e_j) \geq \sum_{j=1}^l (b_j(1)/m_1) e_j$ since $m_1 > 0$. But $b_j(1)/m_1 \geq b_j(i)/m_i$ for all $i = 1, \dots, r$ and $m_i > 0$ hence

$$\langle \omega_i^{-1}, \xi \rangle = 1/m \left(m_i \left(me_0 + \sum_{j=1}^l c_j e_j \right) - \sum_{j=1}^l b_j(i) e_j \right) \geq 0$$

for all $i = 1, \dots, r$.

Suppose conversely that for all $\xi \in \Xi \cap \overline{C}$ if $\langle \omega_1^{-1}, \xi \rangle \geq 0$ then $\langle \omega_i^{-1}, \xi \rangle \geq 0$ for all $i = 1, \dots, r$. Let $\xi = \xi_0^{-m_i m}$. If

$$\langle \omega_1^{-1}, \xi \rangle = 1/m(m_1(-m_i m)) = -m_1 m_i \geq 0$$

then

$$\langle \omega_i^{-1}, \xi \rangle = 1/m(m_i(-m_i m)) = -m_i^2 \geq 0.$$

If $m_1 = 0$ then $m_i = 0$ for all $i = 1, \dots, r$ but this is not the case. If $m_1 > 0$ then $m_i \geq 0$. In fact $m_i > 0$ since otherwise $b_{j_0}(i) \neq 0$ for some j_0 and then setting $e_{j_0} = m$, $e_0 \geq b_{j_0}(1)/m_1 - c_{j_0}e_{j_0}$ and $e_j = 0$ otherwise, we would have $\langle \omega_1^{-1}, \xi \rangle \geq 0$ but $\langle \omega_i^{-1}, \xi \rangle < 0$. A similar argument works for $m_1 < 0$. Finally for each $j = 1, \dots, l$, choose $e_j = m_1 m$ and $e_0 = b_j(1) - c_j$ and $e_i = 0$ otherwise. Then

$$\langle \omega_1^{-1}, \xi \rangle = 1/m(m_1(m(b_j(1) - c_j) + c_j m_1 m) - b_j(1)m_1 m) = 0.$$

Hence our assumption implies that

$$\begin{aligned} \langle \omega_i^{-1}, \xi \rangle &= 1/m(m_i(m(b_j(1) - c_j) + c_j m_1 m) - b_j(i)m_1 m) \\ &= 1/m(m_i m b_j(1) - b_j(i)m_1 m) \geq 0. \end{aligned}$$

Since $m_1 m_i > 0$, we have that $b_j(1)/m_1 \geq b_j(i)/m_i$ for all $i = 1, \dots, r$ i.e., that ${}^t \rho_1^{-1}$ dominates ${}^t \rho_i^{-1}$ for each $i = 1, \dots, r$. This completes the proof. \square

The following corollary of this result describes a natural setting in which Assumption 5.5 is true. Note that the hypothesis of the lemma is true for the example of section 3.1 (see the proof of Lemma 3.2 (3)).

LEMMA 5.7. *Suppose that ρ_1 has the property: (*) for all $g \in G$, $\rho_1(g) \in M_{n_1}(\vartheta_k)$ implies that $\rho_i(g) \in M_{n_i}(\vartheta_k)$ for each $i = 1, \dots, r$. Then ρ_1 dominates ρ_i for each $i = 1, \dots, r$.*

Proof. Let $\xi \in \Xi \cap \overline{C}$ then $\xi(\pi) \in T$. Suppose that $\langle \omega_1^{-1}, \xi \rangle \geq 0$. Then by the proof of Lemma 5.4, $\langle \omega_{1j}, \xi \rangle \geq 0$, for each $j = 1, \dots, n_1$. But this means that $\omega_{1j}(\xi(\pi)) \in \vartheta_k$ for each $j = 1, \dots, n_1$, i.e. that $\rho_1(\xi(\pi)) \in M_{n_1}(\vartheta_k)$. Hence property (*) implies that $\rho_i(\xi(\pi)) \in M_{n_i}(\vartheta_k)$, i.e. that $\langle \omega_{ij}, \xi \rangle \geq 0$, for each $j = 1, \dots, n_i$. In particular $\langle \omega_i^{-1}, \xi \rangle \geq 0$. By Lemma 5.6 this implies that ${}^t \rho_1^{-1}$ dominates ${}^t \rho_i^{-1}$ for each $i = 1, \dots, r$. \square

Under Assumption 5.5 we can then prove

LEMMA 5.8.

$$\Xi^+ \cap \overline{C} = \{ \xi_0^{me_0} \cdot \prod_{j=1}^l (\xi_0^{c_j} \xi_j)^{e_j} \mid e_j \geq 0 \text{ for } 1 \leq j \leq l \text{ and } me_0 \geq \sum_{j=1}^l (b_j(1)/m_1 - c_j)e_j \}.$$

Proof. An element $\xi = \xi_0^{me_0} \cdot \prod_{j=1}^l (\xi_0^{c_j} \xi_j)^{e_j}$ of Ξ is contained in Ξ^+ if and only if $\omega_{ik}(\xi(\pi)) \in \vartheta_k$, i.e. $\langle \omega_{ik}, \xi \rangle \geq 0$, for each $i = 1, \dots, r$ and $k = 1, \dots, n_i$. But by Lemma 5.4 and 5.6, under Assumption 5.5 this is the case if and only if $\langle \omega_1^{-1}, \xi \rangle \geq 0$, i.e. $me_0 \geq \sum_{j=1}^l (b_j(1)/m_1 - c_j)e_j$. \square

We can finally complete our explicit finite form for $Z_{G,\rho,f}(s)$. Since $|f(\xi(\pi))| = q^{-\langle \alpha_0, \xi \rangle}$ for $\xi \in \Xi$, we have by (5.4), (5.6) and Lemma 5.8

$$Z_{G,\rho,f}(s) = \sum_{w \in W} q^{-\lambda(w)} \sum_{(e_0, \dots, e_l) \in I} q^{-me_0s + (a_1 - c_1s)e_1 + \dots + (a_l - c_ls)e_l}$$

where

$$I = \left\{ (e_0, \dots, e_l) \in \mathbf{Z}^l \mid me_0 \geq \sum_{j=1}^l (b_j(1)/m_1 - c_j)e_j, e_j \geq 1 \text{ if } \alpha_j \in w(\Phi^-) \text{ and } e_j \geq 0 \text{ otherwise for } j = 1, \dots, l \right\}.$$

Note that, since $\langle \omega_1^{-1}, \xi \rangle \in \mathbf{Z}$, $1/m \sum_{j=1}^l (b_j(1)/m_1 - c_j)e_j \in \mathbf{Z}$. Hence

$$Z_{G,\rho,f}(s) = \frac{(\sum_{w \in W} q^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} q^{a_j - (b_j(1)/m_1)s})}{(1 - q^{-ms}) \prod_{j=1}^l (1 - q^{a_j - (b_j(1)/m_1)s})}$$

provided that $Re(s) > \max\{0, a_j m_1 / b_j(1) \mid j = 1, \dots, l\}$.

Igusa showed that by choosing a different basis for the root system Φ we can realize a functional equation that $Z_{G,\rho,f}(s)$ satisfies. We prefer to explain this functional equation via a change of variable in this expression. We can write for any $w_0 \in W$

$$Z_{G,\rho,f}(s) = \frac{(\sum_{w' \in W} q^{-\lambda(w'w_0)} \prod_{\alpha_j \in w'w_0(\Phi^-)} q^{a_j - (b_j(1)/m_1)s})}{(1 - q^{-ms}) \prod_{j=1}^l (1 - q^{a_j - (b_j(1)/m_1)s})}.$$

We now choose w_0 to be the unique element of W permuting Φ^+ and Φ^- . Then using the fact that $\lambda(w'w_0) + \lambda(w') = \text{card}(\Phi^+)$ and denoting the above two

expressions for the same $Z_{G,\rho,f}(s)$ by Z^+ and Z^- we get

$$Z^+|_{q \rightarrow q^{-1}} = (-1)^{l+1} q^{-ms + \text{card}(\Phi^+)} Z^-.$$

This is our form of the functional equation established by Igusa.

We draw together the results of this section and section 2 in the following:

THEOREM 5.9. *Let G be a linear algebraic group defined over k , a finite extension of \mathbf{Q}_p and $\rho : G \rightarrow \text{GL}_n$ be a faithful k -rational representation. Let H denote the connected component of the reductive part of G . Suppose that G and ρ satisfy Assumptions 2.1, 2.2 and 2.3 and H and $\rho|_H$ satisfy Assumptions 5.1 to 5.5 and that the functions $\theta_i : H \rightarrow \mathbf{R}$ defined in §2 are characters on H . Then*

(1)

$$Z_{G,\rho}(s) = \frac{(\sum_{w \in W} q^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} q^{a_j - (b_j(1)/m_1)(r_1 s + r_2)})}{(1 - q^{-m(r_1 s + r_2)}) \prod_{j=1}^l (1 - q^{a_j - (b_j(1)/m_1)(r_1 s + r_2)})}$$

provided that $\text{Re}(s) > \max\{-r_2/r_1, 1/r_1(a_j m_1 / b_j(1) - r_2)\}$.

(2) $Z_{G,\rho}(s)$ satisfies a functional equation:

$$Z_{G,\rho}(s)|_{q \rightarrow q^{-1}} = (-1)^{l+1} q^{-m(r_1 s + r_2) + \text{card}(\Phi^+)} Z_{G,\rho}(s).$$

We recall briefly the interpretation of the numerical data m, r_1 and $r_2, a_1, \dots, a_l, b_1(1), \dots, b_l(1)$ and m_1 :

m. Let S denote the one-dimensional maximal central torus of H and H' the derived group of H , then $H' \cap S = \mu_m$, the group of m th roots of unity.

r_1 and r_2 . Let f denote the generator of $\text{Hom}(G, \mathbf{G}_m)$ and $\theta_i (i = 1, \dots, c-1)$ be the functions on H detailed in section 2 (assumed to be characters), then $f^{r_1}(h) = |\det \rho(h)|$ and $f^{r_2}(h) = \prod_{i=1}^{c-1} |\theta_i(h)|^{-1}$.

a_1, \dots, a_l . Let $\{\alpha_1, \dots, \alpha_l\}$ denote a basis for the root system Φ of H relative to a maximal k -split torus T , then $\prod_{\alpha \in \Phi^+} \alpha = \alpha_1^{a_1} \dots \alpha_l^{a_l}$.

$b_1(1), \dots, b_l(1)$ and m_1 . $\rho|_H$ decomposes as a direct sum of irreducible representations ρ_1, \dots, ρ_r where ρ_1 dominates ρ_2, \dots, ρ_r . Let ω_1 denote the dominant weight of ${}^t \rho^{-1}$ then $\omega_1^m = \alpha_0^{-m_1} \cdot \prod_{j=1}^l \alpha_j^{b_j(1)}$ where $\alpha_0 = f|_T$.

6. Functional equations and uniformity for local zeta functions of algebraic groups. We return in this section to the perspective introduced in §4. Let G be a linear algebraic group defined over a number field K and fix a faithful

K -rational representation

$$\rho : G \rightarrow \mathrm{GL}_n.$$

As a corollary to the previous section, under certain conditions we shall deduce for almost all primes \mathfrak{p} of K an explicit finite form and a functional equation for the zeta functions $Z_{G,\rho,\mathfrak{p}}(s)$. A corollary of this explicit finite form will be a certain uniformity in $Z_{G,\rho,\mathfrak{p}}(s)$ as \mathfrak{p} ranges over all primes of K .

In particular, when $G = \mathrm{Aut}\Gamma$ and $K = \mathbf{Q}$ for some torsion-free, finitely generated nilpotent group Γ we can deduce corresponding results for the local zeta functions defined in the Introduction for the nilpotent group Γ .

THEOREM 6.1. *Let G be a linear algebraic group defined over K a finite extension of \mathbf{Q} and $\rho : G \rightarrow \mathrm{GL}_n$ be a faithful K -rational representation. Let H be the connected component of the reductive part of G . Suppose that (i) the functions $\theta_i : H \rightarrow \mathbf{R}$ (defined in §2) are characters of H ; (ii) H is K -split; (iii) the maximal central torus S of H is one-dimensional; and (iv) there exists an irreducible component ρ_1 of ρ which dominates the remaining irreducible components. Then for almost all primes \mathfrak{p} of K :*

(1)

$$Z_{G,\rho,\mathfrak{p}}(s) = \frac{(\sum_{w \in W} q^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} q^{a_j - (b_j(1)/m_1)(r_1s+r_2)})}{(1 - q^{-m(r_1s+r_2)}) \prod_{j=1}^l (1 - q^{a_j - (b_j(1)/m_1)(r_1s+r_2)})}$$

provided that $\mathrm{Re}(s) > \max\{-r_2/r_1, 1/r_1(a_jm_1/b_j(1) - r_2)\}$ where the numerical data m , r_1 , and r_2 , $a_1, \dots, a_l, b_1(1), \dots, b_l(1)$ and m_1 , have the interpretation detailed at the end of §5 and q is the order of the residue field of $K_{\mathfrak{p}}$; and

(2) $Z_{G,\rho,\mathfrak{p}}(s)$ satisfies a functional equation:

$$Z_{G,\rho,\mathfrak{p}}(s)|_{q \rightarrow q^{-1}} = (-1)^{l+1} q^{-m(r_1s+r_2)+\mathrm{card}(\Phi^+)} Z_{G,\rho,\mathfrak{p}}(s).$$

Proof. By Corollary 4.6 for almost all primes \mathfrak{p} of K we can express $Z_{G,\rho,\mathfrak{p}}(s)$ as an integral (4.1) with respect to the reductive group H . We are required to show that, for almost all primes, H and ρ satisfy Assumptions 5.3 and 5.4.

By Lemma 4.2 for any equivalent representation ρ' of ρ , $Z_{G,\rho',\mathfrak{p}}(s) = Z_{G,\rho,\mathfrak{p}}(s)$ for almost all primes \mathfrak{p} . Since H is a reductive group, the representation ρ is completely reducible, i.e., equivalent over $\mathrm{GL}_n(K)$ to a direct sum of irreducible representations ρ_i called the irreducible components of ρ . (The irreducible components are uniquely determined up to equivalence. Note that the concept of a representation ρ_1 dominating a representation ρ_2 is invariant under taking equivalent representations so Assumption (iv) of our Theorem is well-defined.) Since T splits over K , $\rho_i(T(K))$ is diagonalizable over $\mathrm{GL}_n(K)$ so we can choose ρ_i with $\rho_i(T(K))$ diagonal. Hence by taking an equivalent representation we can arrange that, for almost all primes \mathfrak{p} , Assumption 5.4 is true.

If $\pi_{\mathfrak{p}}$ denotes a fixed uniformizing parameter for $\vartheta_{K_{\mathfrak{p}}}$ then, for almost all primes \mathfrak{p} , $H(K_{\mathfrak{p}})$ has very good reduction mod $\pi_{\mathfrak{p}}$ (see for example Proposition 3.20 [PR]).

Since we are taking Assumptions 5.1, 5.2 and 5.5 as hypotheses for our Theorem we can apply Theorem 5.9 to yield statements (1) and (2) above for almost all primes \mathfrak{p} . \square

Note that the numerical data are independent of the prime \mathfrak{p} . (A trap for the unwary: m is not the order of the group $H'(K_{\mathfrak{p}}) \cap S(K_{\mathfrak{p}})$, which depends on \mathfrak{p} , but the order of the group $H'(\overline{K}_{\mathfrak{p}}) \cap S(\overline{K}_{\mathfrak{p}})$, (where $\overline{K}_{\mathfrak{p}} = \mathbf{C}$ is the algebraic closure of $K_{\mathfrak{p}}$) which does not depend on \mathfrak{p}). We thus have the following uniformity result:

COROLLARY 6.2. *Suppose that G and ρ satisfy the hypothesis of Theorem 6.1. Then there exists a rational function $W(X, Y) \in \mathbf{Q}(X, Y)$ such that for almost all primes \mathfrak{p} of K*

$$Z_{G, \rho, \mathfrak{p}}(s) = W(q, q^{-s}),$$

i.e. $Z_{G, \rho}$ is universal in \mathfrak{p} .

It would be desirable to remove the hypothesis made on the reductive group in Theorem 6.1—in particular, that H be K -split. We can however already extend Theorem 6.1 to a class of non-split groups—namely to groups which are the restriction of scalars of a split group over a larger field.

Restriction of scalars for abstract algebraic varieties was defined by Weil [W] and the definition reproduced in the languages of schemes in [BS] §2.8. We follow the construction in [Se] and [PR] §2.1.2 for the special case of an algebraic matrix group since we also want to keep track of restricting the representation $\rho : G \rightarrow \mathrm{GL}_n$. Identify G via ρ with its image as an algebraic subgroup of GL_n . Let L be a finite extension of K of degree d where $d|n$. The construction depends on a choice of K -basis \mathcal{E} for L . We choose \mathcal{E} to be an integral basis. In this way we will ensure that integral matrices restrict to integral matrices. For any extension field E of K we take $R = L \otimes_K E$ to be the E -algebra on the basis \mathcal{E} with the same structure constants as the K -algebra L . R acts by multiplication on itself as an E -algebra and hence we can identify R with an E -subalgebra \mathcal{C}_E of $\mathrm{M}_d(E)$, the E -linear transformations of R .

Let \mathcal{G} be a L -algebraic subgroup of GL_m where $m = n/d$. For any extension field L_1 of K , $\mathcal{G}(L_1)$ is the set of all matrices x in $\mathrm{GL}_m(L_1)$ which satisfy certain polynomial equations $P_l(x_{11}, x_{12}, \dots, x_{mm}) = 0$ ($l \in \Lambda$).

For any extension field E of K , define $\mathcal{R}_{L/K}\mathcal{G}(E)$ to be the set of matrices in $\mathrm{M}_n(E)$ which can be written as $m \times m$ matrices whose entries are themselves matrices belonging to $\mathcal{C}_E \leq \mathrm{M}_d(E)$ and which, considered as $m \times m$ matrices, satisfy the equations defining \mathcal{G} . Since \mathcal{C}_E is defined by polynomials over K , and the equations $P_l = 0$ force \mathcal{G} to be a subgroup of GL_m , $\mathcal{R}_{L/K}\mathcal{G}$ is a K -algebraic

subgroup of GL_n called the restriction of scalars of \mathcal{G} from L to K . The group $\mathcal{R}_{L/K}\mathcal{G}(E)$ may be identified with the group $\mathcal{G}(R)$ where $R = L \otimes_K E$.

Let ϑ_E be the ring of integers of an extension field E .

LEMMA 6.3. $\mathcal{R}_{L/K}\mathcal{G}(E) \cap \mathrm{M}_n(\vartheta_E) = \mathcal{G}(R) \cap \mathrm{M}_m(\vartheta_L \otimes_{\vartheta_K} \vartheta_E)$ where $R = L \otimes_K E$.

Proof. It suffices to prove that

$$\mathcal{C}_E \cap \mathrm{M}_d(\vartheta_E) = \vartheta_L \otimes_{\vartheta_K} \vartheta_E.$$

Let $\mathcal{E} = \{e_1, \dots, e_d\}$ be the integral basis for L over K . Then $\vartheta_L \otimes_{\vartheta_K} \vartheta_E = \vartheta_E \cdot e_1 \oplus \dots \oplus \vartheta_E \cdot e_d$. Since $e_i \cdot e_j \in \vartheta_L$, multiplication by an element $x \in \vartheta_L \otimes_{\vartheta_K} \vartheta_E$ is represented with respect to the basis \mathcal{E} by a matrix in $\mathrm{M}_d(\vartheta_E)$. Hence $\vartheta_L \otimes_{\vartheta_K} \vartheta_E \subseteq \mathcal{C}_E \cap \mathrm{M}_d(\vartheta_E)$. Conversely a matrix $u \in \mathcal{C}_E$ represents multiplication by the element $1.u$. There exist $a_1, \dots, a_d \in \vartheta_K$ with the property that $1 = a_1 e_1 + \dots + a_d e_d$. If $u = (u_{ij}) \in \mathrm{M}_d(\vartheta_E)$ then $1.u = (\sum_{j=1}^d a_j u_{1j})e_1 + \dots + (\sum_{j=1}^d a_j u_{dj})e_d \in \vartheta_L \otimes_{\vartheta_K} \vartheta_E$. Thus $\mathcal{C}_E \cap \mathrm{M}_d(\vartheta_E) \subseteq \vartheta_L \otimes_{\vartheta_K} \vartheta_E$. This completes the proof of Lemma 6.3. \square

LEMMA 6.4. *Suppose that the reductive K -algebraic group G is the restriction of scalars of a reductive L -algebraic group \mathcal{G} . If $\beta \in \mathrm{Hom}_K(G, \mathbf{G}_m)$ is a character of G defined over K then, identifying $G(K)$ with $\mathcal{G}(L)$, β is a character of $\mathcal{G}(L)$ defined over L .*

Proof. Recall that a reductive group is the almost direct product of its maximal central torus and its derived group. Hence β is a character of $\mathcal{G}(L)$ defined over L if and only if $\beta|_S$ is a character of S (where S is the maximal central L -torus of \mathcal{G}) defined over L and $\beta|_{\mathcal{G}'} = \mathrm{id}$ (where \mathcal{G}' defines the derived group of \mathcal{G}). Let $S_0 = \mathcal{R}_{L/K}S$. Then S_0 is the maximal central K -torus of G . This follows from the fact $\mathcal{R}_{L/K}$ induces a one-to-one correspondence between L -subgroups in \mathcal{G} and K -subgroups in G and preserves the properties of being a torus or being central (see [BT] §6.18-19). A proof of our lemma for G a torus is contained in [O] §1.4. Since $\beta|_{S_0}$ is a character of S_0 defined over K , [O] implies that $\beta|_S$ defines a character of S over L . If $V(G)$ is any verbal subgroup of G then $V(G) = \mathcal{R}_{L/K}V(\mathcal{G})$. Thus $G' = \mathcal{R}_{L/K}\mathcal{G}'$ and hence $\beta|_{\mathcal{G}'} = \mathrm{id}$. This proves our lemma. (We also refer to §2.1.2 of [PR] where the statement of this lemma is mentioned.) \square

Having set up the language for restriction of scalars we may now state the following:

PROPOSITION 6.5. *Let G be a linear algebraic group defined over a number field K and $\rho : G \rightarrow \mathrm{GL}_n$ be a faithful K -rational representation. Choose two characters*

$\beta_1, \beta_2 \in \text{Hom}_K(G, \mathbf{G}_m)$ of G . Define, for each prime \mathfrak{p} of K ,

$$Z_{G, \rho, \beta_1, \beta_2, \mathfrak{p}}(s) = \int_{G_{\mathfrak{p}}^+} |\beta_1(g)|^s |\beta_2(g)| \mu_G(g)$$

where $G_{\mathfrak{p}}^+ = \rho^{-1}(\rho(G(K_{\mathfrak{p}})) \cap \mathbf{M}_n(\vartheta_{K_{\mathfrak{p}}}))$. Suppose that the matrix group $\rho(G)$ can be identified with the restriction of scalars of an algebraic matrix group $\mathcal{G} \leq \text{GL}_{n_0}$ from an extension field L to K (where $[L : K] = d$ and $n = n_0 d$). Let $\bar{\rho} : \mathcal{G} \rightarrow \text{GL}_{n_0}$ denote the associated representation of \mathcal{G} and suppose that $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ denote the primes in L dividing \mathfrak{p} . Then

$$Z_{G, \rho, \beta_1, \beta_2, \mathfrak{p}}(s) = \prod_{i=1}^r Z_{\mathcal{G}, \bar{\rho}, \beta_1, \beta_2, \mathfrak{p}_i}(s).$$

Proof. By Lemma 6.3, $\rho(G_{\mathfrak{p}}^+) = \bar{\rho}(\mathcal{G}(L \otimes_K K_{\mathfrak{p}})) \cap \mathbf{M}_{n_0}(\vartheta_L \otimes_{\vartheta_K} \vartheta_{K_{\mathfrak{p}}})$. Since $L \otimes_K K_{\mathfrak{p}} = \prod_{i=1}^r L_{\mathfrak{p}_i}$ and $\vartheta_L \otimes_{\vartheta_K} \vartheta_{K_{\mathfrak{p}}} = \prod_{i=1}^r \vartheta_{L_{\mathfrak{p}_i}}$, it follows that $G_{\mathfrak{p}}^+ = \prod_{i=1}^r \mathcal{G}_{\mathfrak{p}_i}^+$. Lemma 6.4 ensures that the characters β_1 and β_2 define characters of \mathcal{G} over L . Hence

$$Z_{G, \rho, \beta_1, \beta_2, \mathfrak{p}}(s) = \prod_{i=1}^r Z_{\mathcal{G}, \bar{\rho}, \beta_1, \beta_2, \mathfrak{p}_i}(s). \quad \square$$

COROLLARY 6.6. *Let G be a linear algebraic group defined over K a finite extension of \mathbf{Q} and $\rho : G \rightarrow \text{GL}_n$ be a faithful K -rational representation. Let H be the connected component of the reductive part of G . Suppose that $\rho(H)$ can be identified with the restriction of scalars of an algebraic matrix group $\mathcal{H} \leq \text{GL}_{n_0}$ from an extension field L to K (where $[L : K] = d$ and $n = n_0 d$) and that (i) the functions $\theta_i : H \rightarrow \mathbf{R}$ (defined in §2) are characters; (ii) \mathcal{H} is L -split; (iii) the maximal central L -torus S of \mathcal{H} is one-dimensional; and (iv) there exists an irreducible component $\bar{\rho}$ of the natural representation $\bar{\rho}$ of \mathcal{H} which dominates the remaining irreducible components. Then for almost all primes \mathfrak{p} of K :*

(1)

$$(6.1) \quad Z_{G, \rho, \mathfrak{p}}(s) = \prod_{\mathfrak{p}_i \mid \mathfrak{p}} \frac{\left(\sum_{w \in W} q^{-f_i \lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} q^{f_i(a_j - (b_j(1)/m_1)(r_1 s + r_2))} \right)}{\left(1 - q^{-f_i m(r_1 s + r_2)} \right) \prod_{j=i}^l \left(1 - q^{f_i(a_j - (b_j(1)/m_1)(r_1 s + r_2))} \right)}$$

provided that $\text{Re}(s) > \max\{-r_2/r_1, 1/r_1(a_j m_1/b_j(1) - r_2)\}$ where the product $\prod_{\mathfrak{p}_i \mid \mathfrak{p}}$ is taken over all primes \mathfrak{p}_i in L dividing \mathfrak{p} , f_i denotes the residue class degree of \mathfrak{p}_i over \mathfrak{p} and the numerical data m, r_1 , and $r_2, a_1, \dots, a_l, b_1(1), \dots, b_l(1)$ and m_1 is associated with the reductive L -algebraic group \mathcal{H} ; and

(2) $Z_{G,\rho,\mathfrak{p}}(s)$ satisfies a functional equation

$$Z_{G,\rho,\mathfrak{p}}(s)|_{q \rightarrow q^{-1}} = \prod_{\mathfrak{p}_i \mid \mathfrak{p}} (-1)^{l+1} q^{f_i(\text{card } \Phi^+ - m(r_1 s + r_2))} Z_{G,\rho,\mathfrak{p}}(s).$$

Proof. This follows directly from Theorem 6.1 and Proposition 6.5. \square

Note that this corollary encompasses groups whose reductive part is not K -split. From a comment made in the proof of Lemma 6.4, and using the fact that (L -dimension of an L -torus T)=(K -dimension of $\mathcal{R}_{L/K}T$), the maximal central L -torus of \mathcal{H} is one-dimensional if and only if the maximal central K -torus of H is one-dimensional.

As in Corollary 6.2, we can deduce a uniformity result in the situation detailed in Corollary 6.6, in which the form of the local zeta function depends on how the prime \mathfrak{p} behaves in the extension L .

COROLLARY 6.7. *Suppose that G and ρ satisfy the hypothesis of Corollary 6.6. Then for each finite family $\mathbf{f} = (f_1, \dots, f_r)$ of positive integers there is a rational function $W_{\mathbf{f}}(Y, X)$ such that for almost all primes \mathfrak{p} of K*

$$Z_{G,\rho,\mathfrak{p}} = W_{\mathbf{f}}(q, q^{-s})$$

whenever \mathfrak{p} decomposes in L into r primes of residue class degrees f_1, \dots, f_r respectively.

The Cebotarev Density Theorem then gives us:

COROLLARY 6.8. *Suppose that G and ρ satisfy the hypothesis of Corollary 6.6. Then $Z_{G,\rho}(s)$ is almost universal in \mathfrak{p} .*

Finally we draw all this together to conclude the following theorem about the zeta function $\zeta_{\Gamma}^{\wedge}(s)$ associated to a finitely generated torsion-free nilpotent group Γ :

THEOREM 6.9. *Let Γ be a finitely generated torsion-free nilpotent group or a ring additively isomorphic to \mathbf{Z}^d . Suppose that the algebraic automorphism group associated to Γ satisfies the hypothesis of Corollary 6.6. Then:*

- (1) *for almost all primes p , $\zeta_{\Gamma,p}^{\wedge}(s)$ is a rational function of the form (6.1);*
- (2) *$\zeta_{\Gamma}^{\wedge}(s)$ is almost universal in p ;*
- (3) *for almost all primes p , $\zeta_{\Gamma,p}^{\wedge}(s)$ satisfies a functional equation of the form*

$$\zeta_{\Gamma,p}^{\wedge}(s)|_{p \rightarrow p^{-1}} = (-1)^{n_i} p^{a_i s + b_i} \zeta_{\Gamma,p}^{\wedge}(s),$$

where $n_i = (l+1)^r$, $a_i = -mr_1 f$, $b_i = (-mr_2 + \text{card } \Phi^+) f$ where $p = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ in L and $f = f_1 + \cdots + f_r$ where f_i is the residue class degree of \mathfrak{p}_i .

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REFERENCES

- [Ba] S. Bachmuth, Induced automorphisms of free groups and free metabelian groups, *Trans. Amer. Math. Soc.* **122** (1966), 1–17.
- [B] A. Borel, Properties and linear representations of Chevalley groups, *Seminar on Algebraic Groups and Related Finite Groups, Lecture Notes in Math.*, vol. 131, Springer-Verlag, New York, 1970, pp. A1–A55.
- [BS] A. Borel and J.-P. Serre, Théorèmes de finitude en cohomologie galoisienne, *Comment. Math. Helv.* **39** (1964), 111–164.
- [BT] A. Borel and J. Tits, Groupes réductifs, *Inst. Hautes Études Sci. Publ. Math.* **27** (1965), 55–151.
- [BG] R. M. Bryant and J. R. J. Groves, Algebraic groups of automorphisms of nilpotent groups and Lie algebras, *J. London Math. Soc.* **33** (1986), 453–466.
- [D] J. Denef, The rationality of the Poincaré series associated to the p -adic points on a variety, *Invent. Math.* **77** (1984), 1–23.
- [DvdD] J. Denef and L. van den Dries, p -adic and real subanalytic sets, *Ann. of Math.* **128** (1988), 79–138.
- [De] M. Deuring, *Algebren*, Second Edition, Springer-Verlag, New York, 1968.
- [duS1] M. P. F. du Sautoy, Finitely generated groups, p -adic analytic groups and Poincaré series, *Bull. Amer. Math. Soc.* **23** (1990), 121–126.
- [duS2] —————, Finitely generated groups, p -adic analytic groups and Poincaré series, *Ann. of Math.* **137** (1993), 639–670.
- [duS3] —————, Counting congruence subgroups in arithmetic subgroups, *Bull. London Math. Soc.* **26** (1994), 255–262.
- [duS4] —————, Zeta functions of groups and rings: uniformity, *Israel J. Math.* **86** (1994), 1–23.
- [duS5] —————, Zeta functions of groups: a survey, preprint.
- [duS6] —————, Functional equations and uniformity for local zeta functions of algebraic groups, in preparation.
- [GSS] F. J. Grunewald, D. Segal and G. C. Smith, Subgroups of finite index in nilpotent groups, *Invent. Math.* **93** (1988), 185–223.
- [H] P. Hall, Nilpotent groups, Queen Mary College Math. Notes, 1969.
- [He] K. Hey, Analytische Zahlentheorie in systemen hyperkomplexer Zahlen, Ph. D. thesis, Hamburg, 1929.
- [I] J.-I. Igusa, Universal p -adic zeta functions and their functional equations, *Amer. J. Math.* **111** (1989), 671–716.
- [II] I. Ilani, Analytic pro- p groups and their lie algebras, *J. Algebra* **176** (1995), 34–58.
- [IwM] N. Iwahori and H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of p -adic Chevalley groups, *Inst. Hautes Études Sci. Publ. Math.* **25** (1965), 5–48.
- [M] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.
- [MKS] W. Magnus, A. Karras and D. Solitar, *Combinatorial Group Theory*, Wiley, Chichester, U. K., 1966.
- [N] L. Nachbin, *The Haar Integral*, Van Nostrand and Co., Princeton, NJ, 1965.

- [O] T. Ono, Arithmetic of algebraic tori, *Ann. of Math.* **74** (1961), 101–139.
- [PR] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, Boston, 1994.
- [Sa] I. Satake, Theory of spherical functions on reductive algebraic groups over p -adic fields, *Inst. Hautes Études Sci. Publ. Math.* **18** (1963), 5–70.
- [Se] D. Segal, On the automorphism groups of certain Lie algebras, *Math. Proc. Cambridge Philos. Soc.* **106** (1989), 67–76.
- [S] J.-P. Serre, *Complex Semisimple Lie Algebras*, Springer-Verlag, New York, 1987.
- [T] T. Tamagawa, On the ζ -function of a division algebra, *Ann. of Math.* **77** (1963), 387–405.
- [W] A. Weil, *Adèles and Algebraic Groups*, Birkhäuser-Verlag, Boston, 1982.
- [Z] M. Zorn, Note zur analytischen hyperkomplexen Zahlentheorie, *Abh. math. Semin. Univ. Hamburg* **9** (1933), 197–201.